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# A sampling theorem for narrowband signals

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**Summary**

The present paper is concerned with a sampling theorem for narrowband signals $s(t)$ (relatively small bandwidth $B$ centered around a high frequency $f_0$). The theorem is based on equidistant samples and the sampling rate can be close to the minimum value of $2B$ samples/sec. Contrary to Woodward's sampling theorem, there is no need for sampling an auxiliary signal derived from $s(t)$.

Some experimental results are presented concerning the reconstruction of the original signal from its samples.

The sampling formula assumes the usual form of a series expansion of $s(t)$ in elementary functions. It is shown that these functions possess the sifting property with respect to the narrowband signal. This property allows any linear operation on the continuous signal $s(t)$ to be replaced by an equivalent operation on the samples, as is desirable for the design of digital signal processing equipment.

## 1. *Introduction*

The subject of the present paper is the equidistant sampling of a narrowband signal $s(t)$. By „narrowband" we mean a signal with a relatively narrow spectral width $B$, centered around a relatively high center frequency $f_0$. More precisely, the signal spectrum $S(f)$ is defined by: **)

$$\left.\begin{aligned} s(t) &= \int S(f) \exp 2\pi \, \mathrm{j}ft \, \mathrm{d}f, \\ S(f) &= \int s(t) \exp - 2\pi \mathrm{j}ft \, \mathrm{d}t, \end{aligned}\right\} \text{ or } s(t) \doteqdot S(f), \tag{1}$$

and we impose the condition that

$$S(f) = 0 \text{ if } f_0 - \tfrac{1}{2} B < |f| < f_0 + \tfrac{1}{2} B, \tag{2a}$$

where $f_0 \geqslant 1\tfrac{1}{2} B$. (2b)

The significance of condition (2b) will be made clear in section 4. The

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**) Integrals or sums without limits indicate integration or summation from $-\infty$ to $+\infty$.

behaviour of $S(f)$ inside the band $B$ is of no concern and is therefore left unspecified. Signals satisfying eq. (2) will be called „narrowband signals". Other signals are labeled "wideband signals". A narrowband signal spectrum is shown in fig. 1.

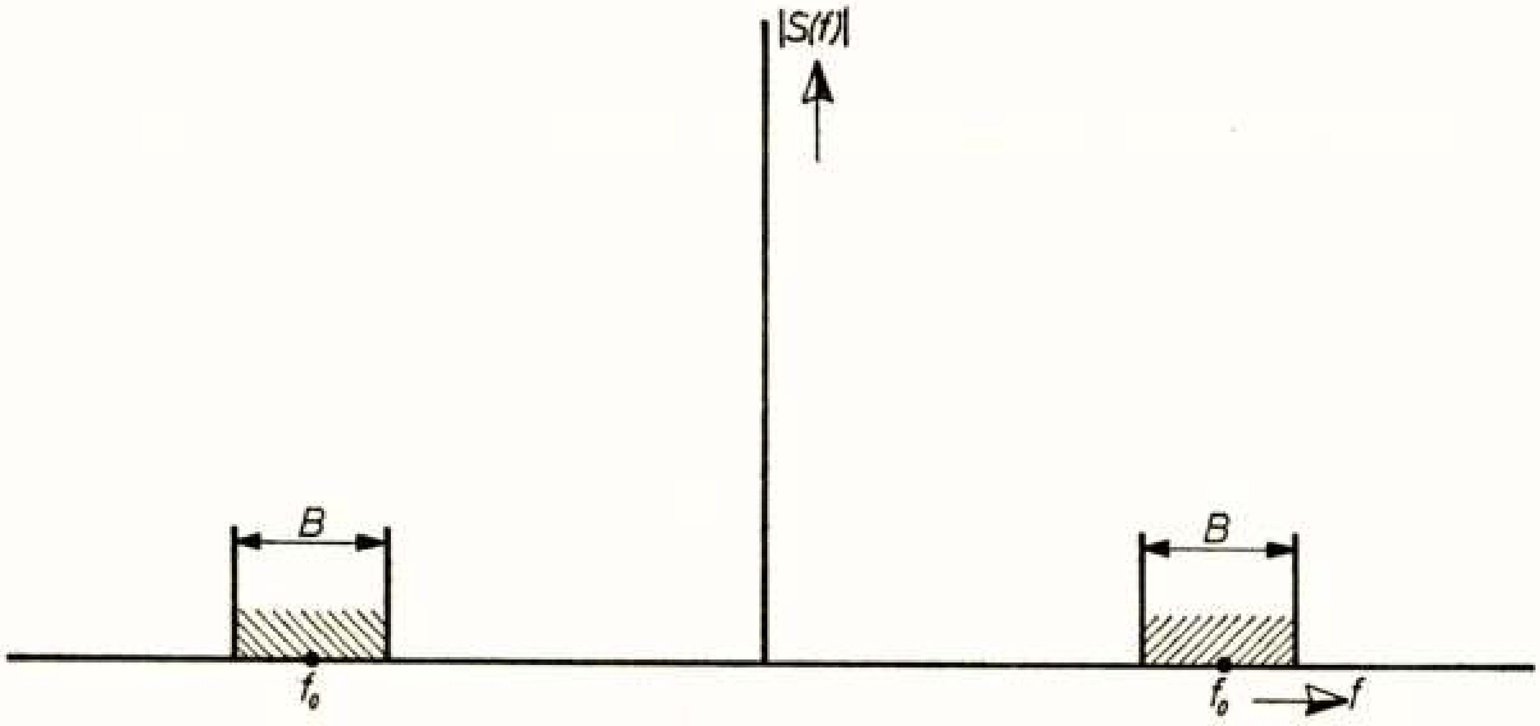



Fig. 1
Spectrum of a narrowband signal

Examples of narrowband signals are radio communication signals, telemetry signals, radar and sonar signals, carrier telephone signals, etc., all at high frequency or intermediate frequency level. Exceptions are voice and music signals at acoustic frequency level, and communication signals at video level.

The purpose of this study is to derive a sampling theorem for narrowband signals with the following properties:

1. Equidistant samples are used at a rate close to the theoretical minimum rate of $2B$ samples/sec.

2. The original signal is reconstructed from its samples by using an expression of the type

$$s(t) = \sum_k s(t_k)\, e\,(t - t_k), \text{ where } t_k = \frac{k}{F} \quad . \tag{3}$$

Here, $s(t_k)$ are samples of the original signal $s(t)$, taken at a rate of $F$ samples/sec, where $F \geqq 2B$. The function $e(t)$ is an elementary function used in the reconstruction process. Shifting this function to the sampling times $t_k$, multiplying with the corresponding sample value and adding yields the original signal.

Physically, eq. 3 represents the output of a linear system with impulse response $e(t)$, which is excited at the input by the sequence of samples

$s(t_k)$ (fig. 2). Mathematically, eq. (3) is a series expansion with the perticular property that the elementary functions $e(t-t_k)$ are time shifted copies of each other and that the expansion coefficients equal the momentaneous signal values.

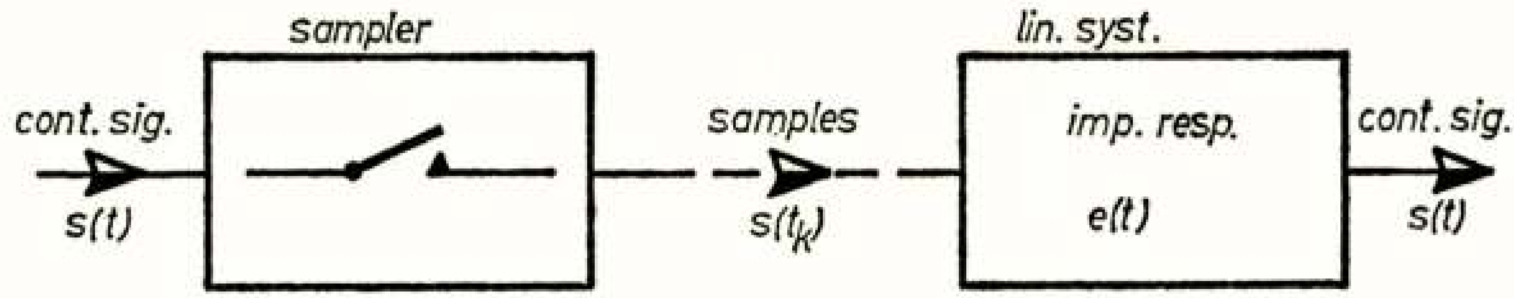



Fig. 2
Sampling and reconstruction of signal (eq.3)

Applications of the narrowband sampling theorem are in those fields where it is desirable to represent a narrowband signal by a series of discrete values. A simple example is the storing of a signal in some type of digital memory. Another application is in the design of signal processing equipment for narrowband signals. In modern communication techniques the required processing can be of such a degree of complexity that the systems can no longer be realized by the use of analog methods. Therefore one has to follow a digital approach and the receivers assume the shape of small special purpose computers. A digital system of this type operates on a sequence of signal samples, rather than on the original continuous signal. It will be clear that a simple and efficient sampling method has to be available in order to keep the required memory capacity at a minimum.

Theoretically, it is possible to heterodyne the original narrowband signal down to the frequency interval from D.C. to $B$ Hz and apply Shannon's sampling theorem to the resulting wideband signal. This amounts to the use of a very low intermediate frequency. In many cases, however, this method leads to a severe complication of the electronics, involved in the signal processing procedure. In other cases (e.g. sonar), there is no need at all to operate at an intermediate frequency because the original signal frequency is already very low. In these cases, the narrowband sampling theorem allows to avoid a reduntant heterodyning stage.

## 2. *Some previous results* *Sampling theorems of Shannon and Woodward*

In order to place the present sampling method in the right perspective it is useful to recalitulate a couple of well known equidistant sampling methods. These will be labeled "Shannon's sampling theorem" and "Woodward's sampling theorem", respectively.

Shannon's sampling theorem (ref. 1) states that any signal $x(t)$ can be represented by a series of samples, taken at minimum rate equalling twice the highest frequency present in the signal spectrum $X(f)$. A reconstruction of the signal is obtained with an expression of the type of eq. (3):

$$\left.\begin{array}{l}\text{If } X(f) = 0 \text{ for } |f| > W, \text{ then} \\ x(t) = \sum_k x(t_k)\, d(t - t_k),\end{array}\right\} \qquad (4)$$

with $t_k = \dfrac{k}{2W}$ and $d(t) = \dfrac{\sin 2\pi Wt}{2\pi Wt}$ = elementary function.

The corresponding block diagram is the same as in fig. 2, with $s(t)$, $e(t)$ replaced by $x(t)$, $d(t)$ respectively. This method is oriented primarily towards wideband signals. In particular, if $X(f)$ exists from D.C. up to a maximum frequency $W$, the theorem yields a sampling rate equal to twice the signal bandwith. As the intrinsic number of degrees of freedom of a signal is determined by its bandwidth, we expect a similar result for a narrowband signal. However, Shannon's theorem is very inefficient when applied to a narrowband signal as it prescribes a sampling rate $F = 2W = 2\,(f_0 + \tfrac{1}{2} B)$. Depending on the value of $f_0$, this can be a very high rate compared to $2B$ samples/sec.

A solution was given by Woodward's sampling theorem (ref. 2), stating that a narrowband signal* $s(t)$ can be written as:

$$s(t) = \sum_k s(t_k)\, a(t - t_k) - \sum_k \hat{s}(t_k)\, b(t - t_k),$$

$$\left.\begin{array}{ll}\text{with } t_k = \dfrac{k}{B} \text{ and } a(t) & = \dfrac{\sin \pi Bt}{\pi Bt} \cos 2\pi f_0 t, \\ \\ b(t) & = \dfrac{\sin \pi Bt}{\pi Bt} \cos 2\pi f_0 t.\end{array}\right\} \qquad (5)$$

Here, $\hat{s}(t)$ is the quadrature signal (Hilbert transform) of $s(t)$. The functions $a(t)$, $b(t)$ are two elementary functions. According to (5) the signal is represented by $B$ samples per second of both $s(t)$ and $\hat{s}(t)$. The total number of samples per second equals $2B$ and the system operates at a

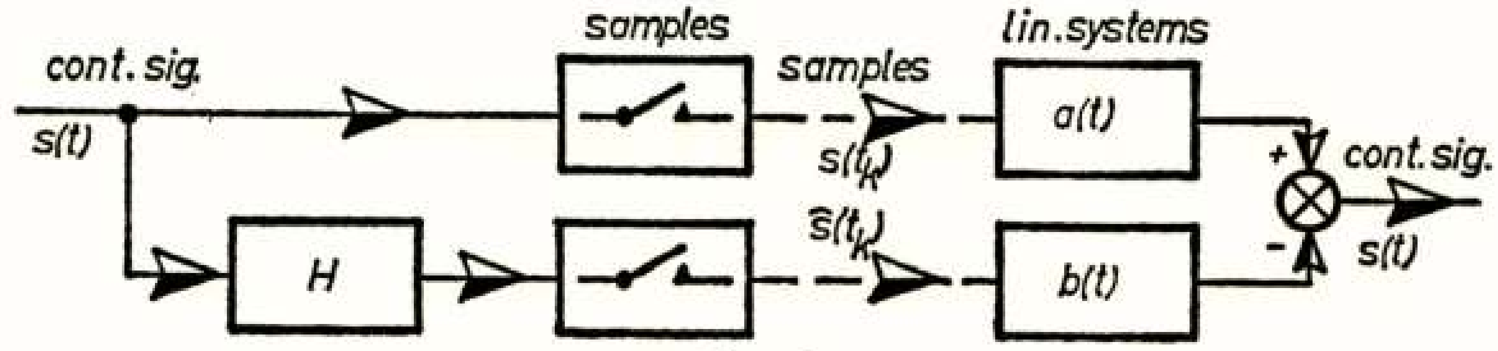



Fig. 3

Sampling and reconstruction according to Woodward's theorem

* Actually, eq. (5) is valid for any signal, although it was designed for narrowband signals.

minimum rate. A block diagram is shown in fig. 3.

From an engineering point of view, the main disadvantage of eq. (5) is that the quardrature signal $\hat{s}(t)$ is not available, in general. Hence it has to be generated from $s(t)$. This requires a frequency independent 90° phase shifter (indicated „H" in fig. 3), or some equivalent system. (For very small $B$, compared with $f_0$, the samples $\hat{s}(t_k)$ can also be obtained approximately by sampling $s(t)$ at time $t_k + \frac{1}{4f_0}$.) Furthermore, the samples of $s(t)$ and $\hat{s}(t)$ have to be stored or processed separately, and also there is a need for two linear systems instead of one when the signal is to be reconstructed from the samples.

The disadvantages of both sampling methods are overcome by the system described below, in which only samples of the original signal $s(t)$ are used at a rate close to $2B$ samples/sec.

3. *Equidistant sampling of a narrowband signal*
*The sampling theorem*

Consider a narrowband signal in the sense of section 1 (bandwidth $B$, centerfrequency $f_0$, $f_0 \geqslant 1\frac{1}{2}\, B$), and let the signal be sampled at a rate of $F$ samples/sec. Representing each sample as a Dirac delta function * the resulting sequence of samples $s'(t)$ is written:

$$s'(t) = \frac{1}{F} \sum_k s(t_k)\, \delta\,(t - t_k), \text{ with } t_k = \frac{k}{F}\,. \tag{6}$$

Now, the problem is to find the spectrum $S'(f)$ of $s'(t)$. Once $S'(f)$ has been derived, the desired expression (eq. (3)) can easily be written down. It can be shown (ref. 2) that the result is:

$$S'(f) = \sum_k S(f) - kF). \tag{7}$$

Hence $S'(f)$ consists of an infinite number of "images" of the original spectral bands, shifted over multiples of $F$ or, in other words, the spectrum of the sample sequence is a periodic repetition of the original spectrum $S(f)$, with a repetition period equal to the sampling rate. Eq. (7) is illustrated in fig. 4b. Note that the original bands around $\pm f_0$ are still present in $S'(f)$.

For a proof of eq. (7) we refer to ref. 2. Qualitatively, eq. (7) can be understood by imagining $s'(t)$ to arise from a multiplication of the original signal $s(t)$ and periodic sequence of delta peaks:

$$s'(t) = s(t)\,.\,r(t), \text{ with } r(t) = \frac{1}{F} \sum_k \delta\,(t - t_k)\,. \tag{8}$$

* Actually, the sample pulses have a small but finite width and an amplitude equal to $s(t_k)$. For theoretical purposes, they are approached by delta peaks without loss of generality.

$r(t)$ consists of an infinite number of equally strong harmonics, at frequencies $kF$; $k = 0, \pm 1, \pm 2, \ldots$ Multiplication of $s(t)$ with each of these harmonics results in sum- and difference-frequencies or, more precisely, in frequency shifted „images" of the original spectrum.

Let us now select a sampling rate $F$ such that none of the repetitions of $S(f)$ (i.e. the terms with $k \neq 0$ in eq. (7)) overlaps the original bands around $\pm f_0$ (i.e. the term with $k = 0$). Assuming the two repetitions adjacent to the original positive frequency band to have indices $k = m$ and $k = m + 1$ (fig. 4b), this condition is satisfied if

$$\left.\begin{aligned}\{-f_0 + mF\} + \tfrac{1}{2} B &\leq f_0 - \tfrac{1}{2} B,\\ \{-f_0 + (m+1) F\} - \tfrac{1}{2} B &\geq f_0 + \tfrac{1}{2} B,\end{aligned}\right\} \text{ or, if} \begin{cases} mF \leq 2 f_0 - B, \\ (m+1) F \geq 2 f_0 + B. \end{cases} \tag{9}$$

The consequences of eq. (9) are investigated in section 4.

Under these conditions, it is evident that $s(t)$ can be reconstructed from its samples by using a bandpass filter which leaves the original bands around $\pm f_0$ unaffected, but which completely suppresses all other spectral components of $S'(f)$. The frequency transfer function of this filter, $H(f)$, is specified by:

$$\begin{aligned} H(f) &= 1 \text{ if } S(f) \neq 0, \\ &= 0 \text{ if } S(f - kF) \neq 0;\ k = \pm 1, \pm 2, \ldots \end{aligned} \tag{10}$$

One example of such a transfer function is illustrated by fig. 4c. Multiplication of eqs. (8) and (10) yields the filter output spectrum:

$$H(f) \,.\, S'(f) = S(f). \tag{11}$$

Thus, the filter output equals the original signal.

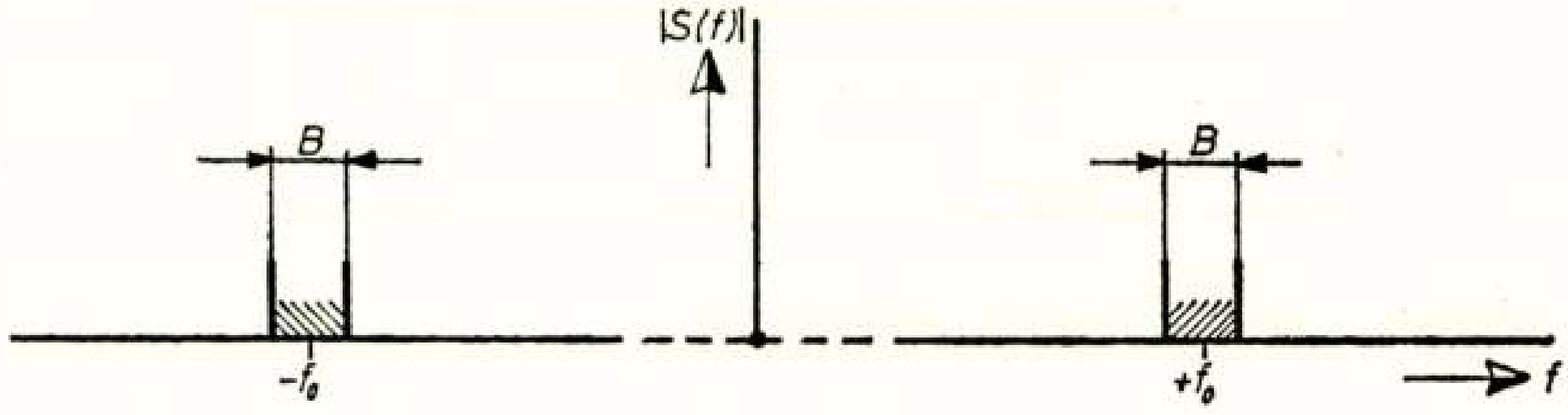
The desired sampling theorem is obtained by transforming eq. (11) to the time domain. Defining $h(t)$ as the impulse response of the reconstruction filter, we get for the transform ef eq. (11) the convolution of $h(t)$ and $s'(t)$:

$$\begin{aligned} s(t) &= \int h(t-\Theta)\, s'(\Theta)\, \mathrm{d}\Theta = \\ &= \frac{1}{F} \int h(t-\Theta) \,.\, \sum_k s(t_k)\, \delta(\Theta - t_k)\, \mathrm{d}\Theta = \frac{1}{F} \sum_k s(t_k)\, h(t - t_k), \end{aligned} \tag{12}$$

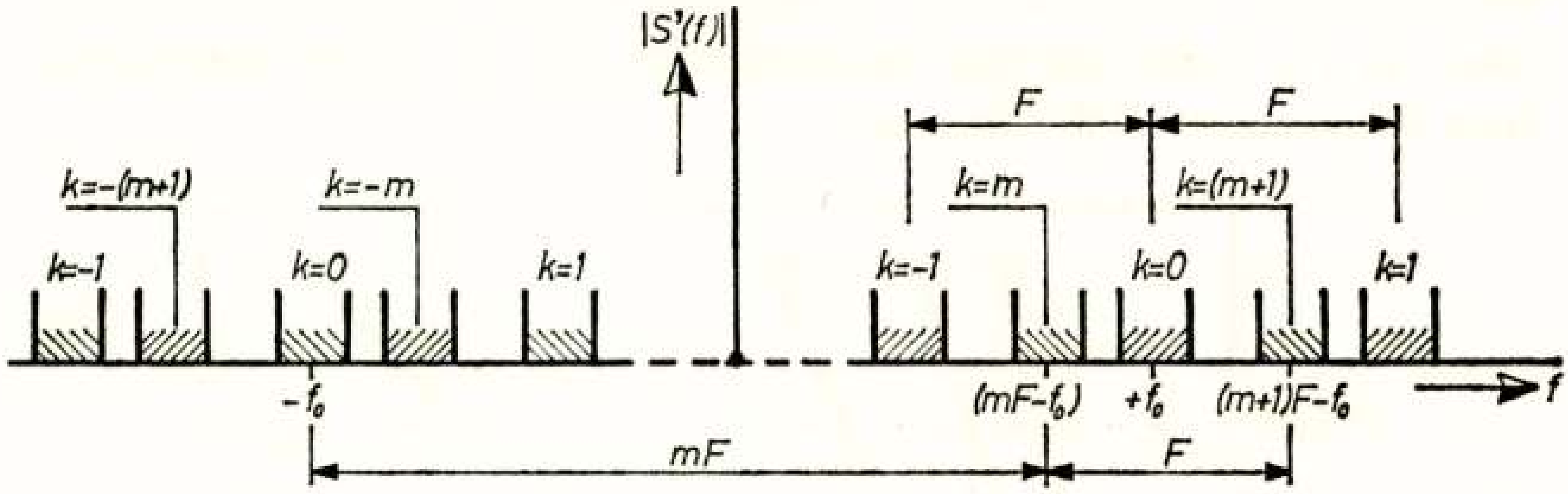
which reflects the fact that the filter output equals the sum of the individual responses to each of the input sample pulses. Finally, we define the elementary function to be proportional to the filter impulse response in order to remove the factor $F$:

$$e(t) = \frac{1}{F}\, h(t) \doteqdot E(f) = \frac{1}{F}\, H(f). \tag{13}$$

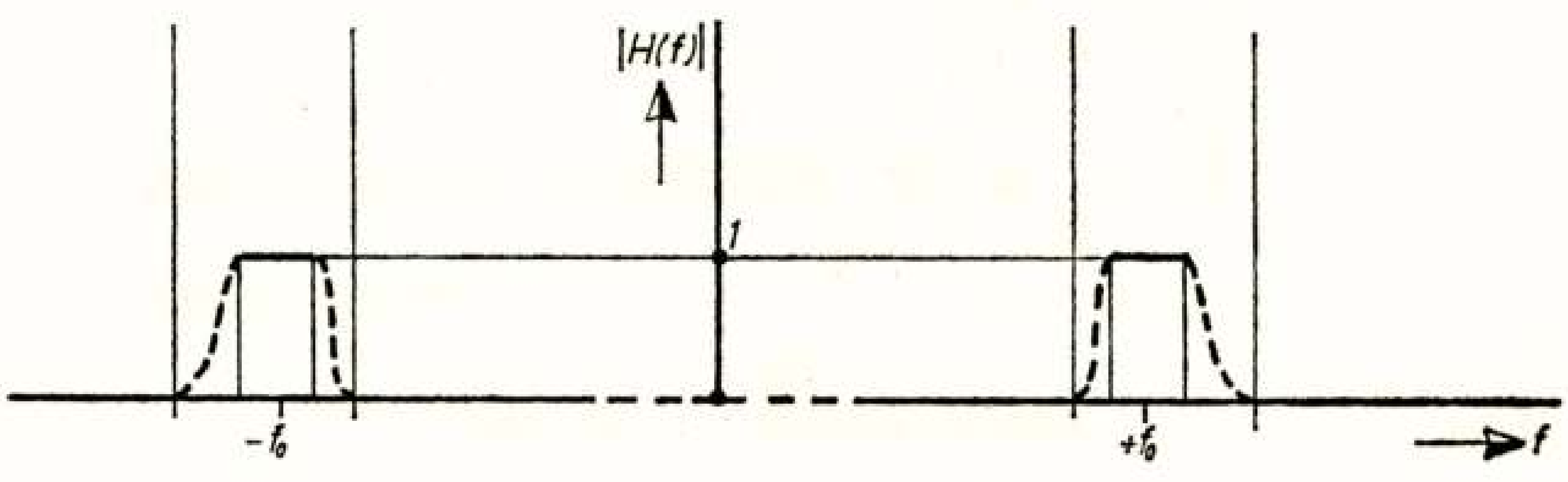
Substitution into eq. (12) gives:




a: Spectrum of continuous signal $s(t)$




b: Spectrum of sampled signal $s'(t)$ (sampling rate $F$)




c: Amplitude transfer function of reconstruction filter

Fig. 4

$$\left.\begin{array}{l} s(t) = \sum\limits_k s(t_k)\, e(t - t_k), \text{ with } t_k = \dfrac{k}{F}\,. \\ F \text{ must satisfy eqs. (9).} \end{array}\right\} \tag{14}$$

4. *Admitted and forbidden sampling rates*

According to section 3, a sampling rate $F$ is admitted if an integer m exists, such that $F$ and m satisfy eqs. (9):

$$mF \leq 2\,f_0 - B = Q\,;\; (m + 1)\,F \geq 2\,f_0 + B = P\,. \tag{15}$$

Note that a minimum condition for eq. (15) is $F \geq 2\,B$. Thus, the sampling rate equals or exceeds the minimum theoretical rate.

Conditions (15) give rise to a number of "admitted intervals" on which the value of $F$ must be chosen. In between we find the „forbidden intervals". In fig. 5 the admitted and forbidden intervals follow immediately from the geometry of the diagram.

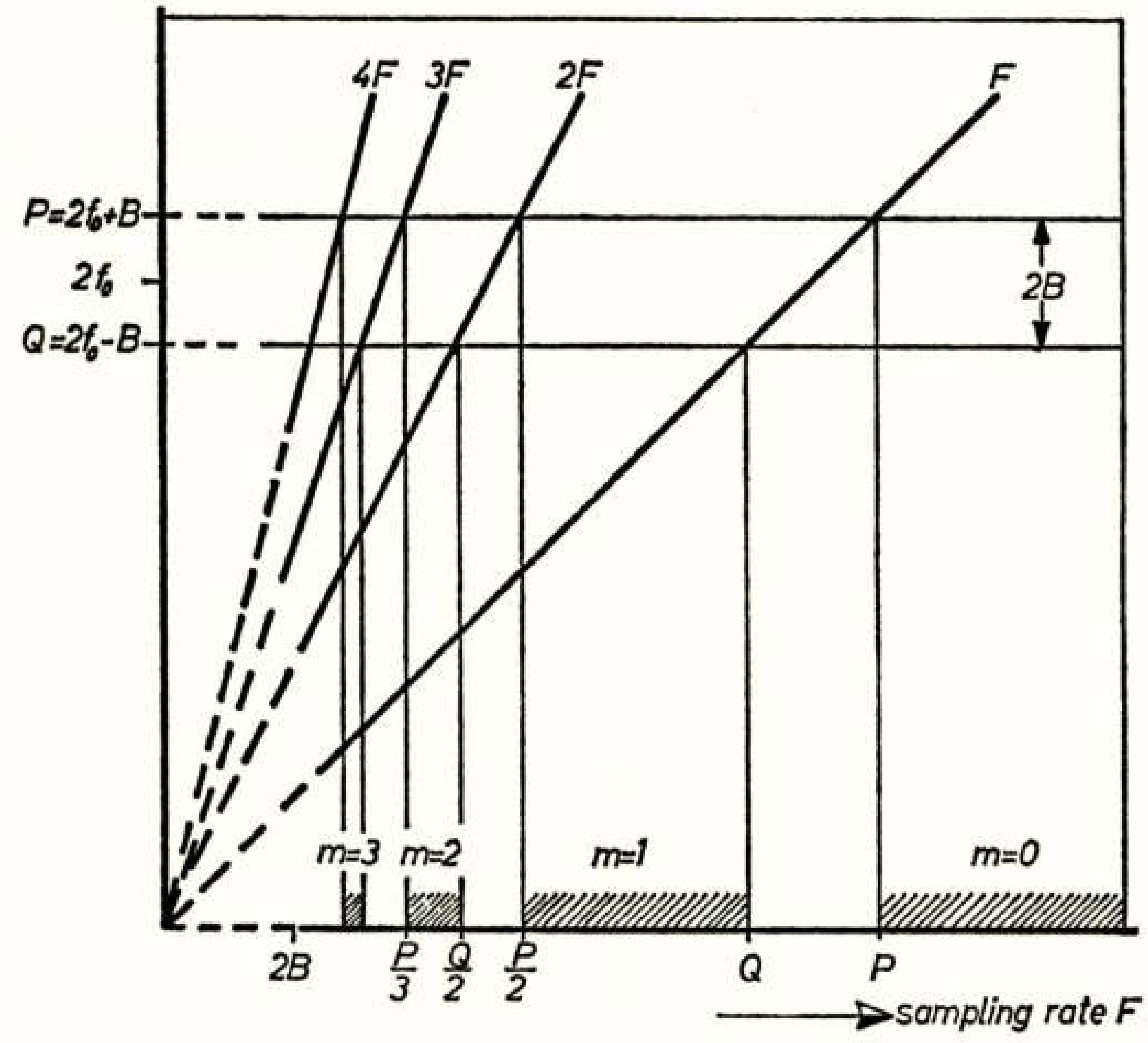



Fig. 5
Admitted sampling rates (shadowed)

The number of admitted intervals is finite. The last interval, containing the highest $F$ values, extends from $F = P$ to $F = \infty$. The first interval, where the smallest values of $F$ are found, is determined by the largest (integral) value of m for which:

$$\frac{P}{m+1} \leqslant \frac{Q}{m} \text{ or, for which } m \leqslant \frac{f_0}{B} - ½\,. \tag{16}$$

Let this largest value be m = M.

$$M = \left(\frac{f_0}{B} - \tfrac{1}{2}\right) - \varepsilon, \text{ with } 0 < \varepsilon < 1\,. \tag{17}$$

Then, there are M + 1 admitted intervals, situated on the $F$ axis as indicated in the table below.

| index of admitted interval | value of m | limits of interval |
|---|---|---|
| 1 | m = M | $\frac{P}{M+1} \leq F \leq \frac{Q}{M}$ |
| 2 | m = M − 1 | $\frac{P}{M} \leq F \leq \frac{Q}{M-1}$ |
| . | . | . |
| . | . | . |
| . | . | . |
| M | m = 1 | $\frac{P}{2} \leq F \leq Q$ |
| (M + 1) | m = 0 | $P \leq F \leq \infty$ |

It is easily verified that already in the first interval $F > 2B$, unless $\frac{f_0}{B}$ — ½ happens to be an integer (i.e. $\varepsilon = 0$ in eq. (17)). In this particular case the minimum sampling rate equals $2B$ samples/sec.

In general, though, the minimum rate will be somewhat higher than $2B$. This is expressed by an inefficiency factor:

$$F_{min} = \frac{P}{M+1} = (1 + \eta)\,.\,2B\,. \tag{18}$$

Substitution of eqs. (15), (17) yields

$$\eta = \frac{(2\,f_0 + B)}{(2\,f_0 + B) - 2\,B\varepsilon} - 1, \text{ with } 0 < \varepsilon < 1\,. \tag{19}$$

Hence $\eta$ rapidly decreases to zero when the ratio $\frac{f_0}{B}$ increases.

For example, $f_0 = 5125$ Hz and $B = 270$ Hz. We then find for the first admitted interval: $553.7 < F < 554.4$. Hence the minimum sampling rate is 553.7 samples/sec. Comparison with $2B = 540$ samples/sec yields an inefficiency factor $\eta = 0.025 = 2.5\%$.

With respect to the forbidden intervals a remark can be made. According to Fourier analysis the narrowband signal can be considered to consist of a large number of sine waves with frequencies inside the bandwith $B$. Suppose, now, that the value of $F$ is such that one of these sine waves (frequency $\nu$) is sampled only at its zero crossings. In this case the signal samples contain no information about this particular signal component. This happens when $F = 2\nu, \nu, \frac{2}{3}\nu, \frac{2}{4}\nu, \frac{2}{5}\nu, \ldots$ Fig. 5 shows that these dangerous sampling rates correspond precisely with the rates inside the forbidden intervals.

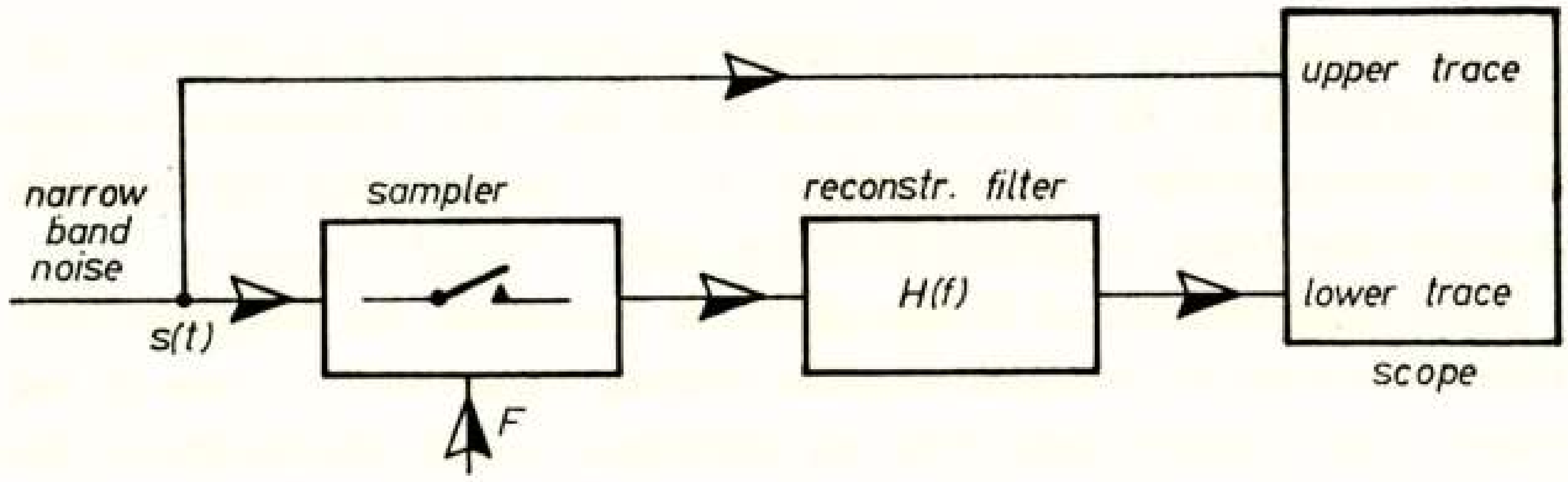
Finally, the restriction $f_0 \geqq 1\frac{1}{2}\, B$ in our definition of a narrowband signal can be made clear (eq. 2b). The present sampling method differs from Shannon's in that the "hole" between the positive and negative frequency band of $s(t)$ is used for accomodating one or more of the spectral images which arise in the sampling process. In this way relatively low sampling rates are obtained. This possibility does not exist, however, when the hole becomes too narrow. It is easily verified that the limit is given by the condition of eq. (2b).

## 5. *Experimental results*
### *Signal restoration*

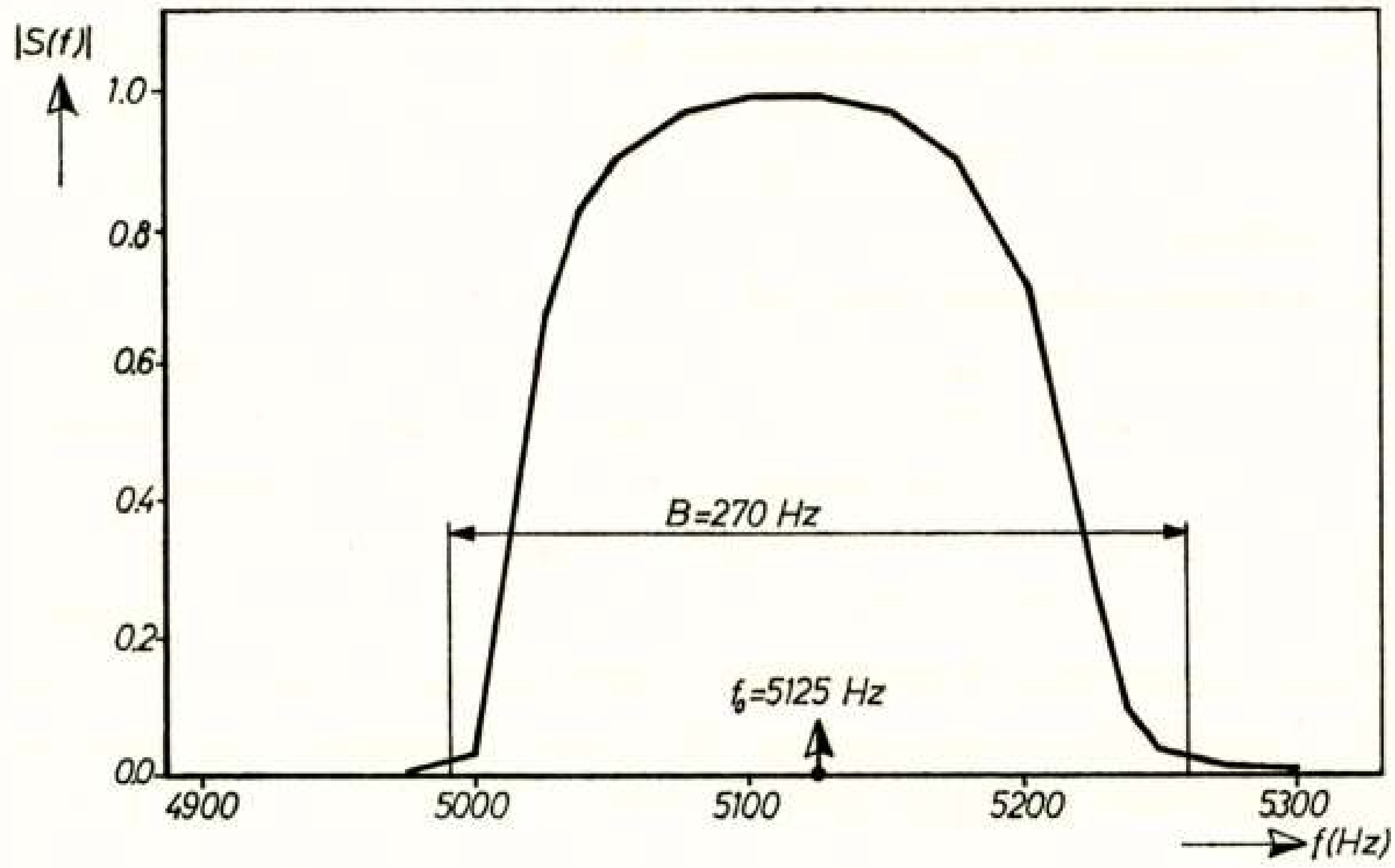
For experimental purposes, eq.(14) is readily implemented. The only problem is the choice of a suitable reconstruction filter. The filter is not completely specified by eq.(10). Its behaviour is left free in the holes between the occupied frequency regions of $S'(f)$.

No advantage can be gained from this freedom, however, and the easiest way is to use one of the usual types of bandpass filter. Clearly, the filter must be physically realizable. This implies that we can not realize a filter with $H(f) = 1$ over the pass band $(f_0 \pm \frac{1}{2}\, B)$, as in eq.(10). Rather, we must aim at a filter with a transfer function given by $H(F) = \exp - 2\pi\, jf\, \tau$. This filter has a linear phase characteristic corresponding to a fixed delay of $\tau$ sec. As a result, the output of the reconstruction filter will be a delayed version of the original signal $s(t)$.

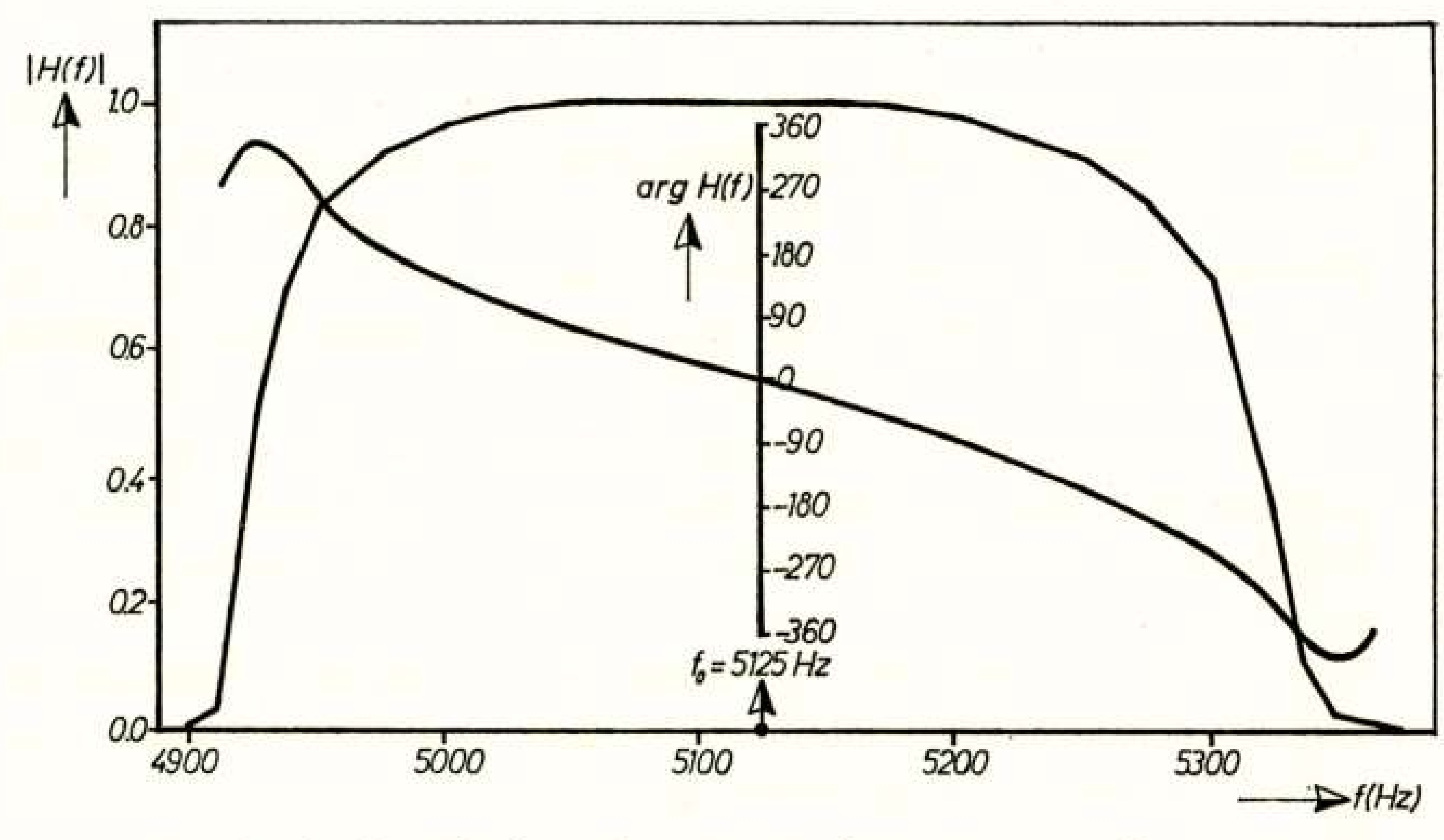
When trying to operate close to the minimum admitted sampling frequency, the requirements on the filter can become prohibitive. The amplitude and phase characteristics over the pass band must be reasonably flat, resp. linear, in order to avoid linear distortion of the signal. On




a: Block diagram of experiment




b: Amplitude density spectrum of narrowband signal $s(t)$




c: Amplitude and phase characteristic of reconstruction filter

Fig. 6

the other hand, the filter skirts must be steep enough to fit into the holes adjacent to the original band (fig. 4b, 4c). Otherwise, a part of the energy in the $k = m$ and $k = m + 1$ images (fig. 4b) will pass through the filter, resulting in an unwanted signal component.

These difficulties can be mitigated by increasing the sampling rate. This widens the holes adjacent to the original band, which results in less severe filter requirements. (As an additional result, the width of the admitted $F$ intervals will decrease and the table of section 4 has to be adapted to the new situation.)

Some results are shown in figures 7, 8, 9, 10, 11, 12, 13. The signal $s(t)$ is narrowband gaussian noise with an amplitude density spectrum as depicted in fig. 6b. Signal bandwith: $B = 270$ Hz; center frequency: $f_0 = 5125$ Hz.

The minimum admitted sampling rate for this case has already been derived in section 4: $F_{min} = 553.7$ samples/sec. As explained above, the practical sampling rate has to be chosen somewhat higher in order to obtain a reasonable reconstruction filter. A suitable compromise was reached by using a reconstruction filter with the characteristics of fig. 6c (bandwith 450 Hz) and a sampling rate $F = 760$ samples/sec. Actually, this is the center value of an admitted interval given by $759 < F < 761$.

Fig. 7 shows the results at $F = 760$. Each photograph contains two input/output pairs. The upper trace represents the original signal and the lower trace the reconstructed one. The delay between upper and lower trace corresponds to the group delay of the reconstruction filter. As these pictures were made at the center of the admitted interval, the results are nearly perfect.

When the value of $F$ is decreased or increased, we expect the resemblance between input and output signal to disappear gradually. This is illustrated in figures 8, 9, 10, 11, 12 and 13. It may be observed that the resemblance is maintained over a wider interval than indicated by the limits given above. This is due to a somewhat conservative estimation of these limits.

A detailed study, using various bandwidths for the signal and the reconstruction filter, has shown that correlation coefficients of 0.97 – 0.99 between original and reconstructed signal can be obtained in practice (ref. 3).

It will be clear from fig. 4 that we are not obliged to filter out the original spectral bands around $\pm f_0$. If desired, the reconstruction filter can be centered around any of the other spectral images present in $S'(f)$. The result will be a reconstructed signal on a different carrier frequency, provided that we do not interchange the images of the original positive and negative frequency bands. An example is shown in fig. 14, where

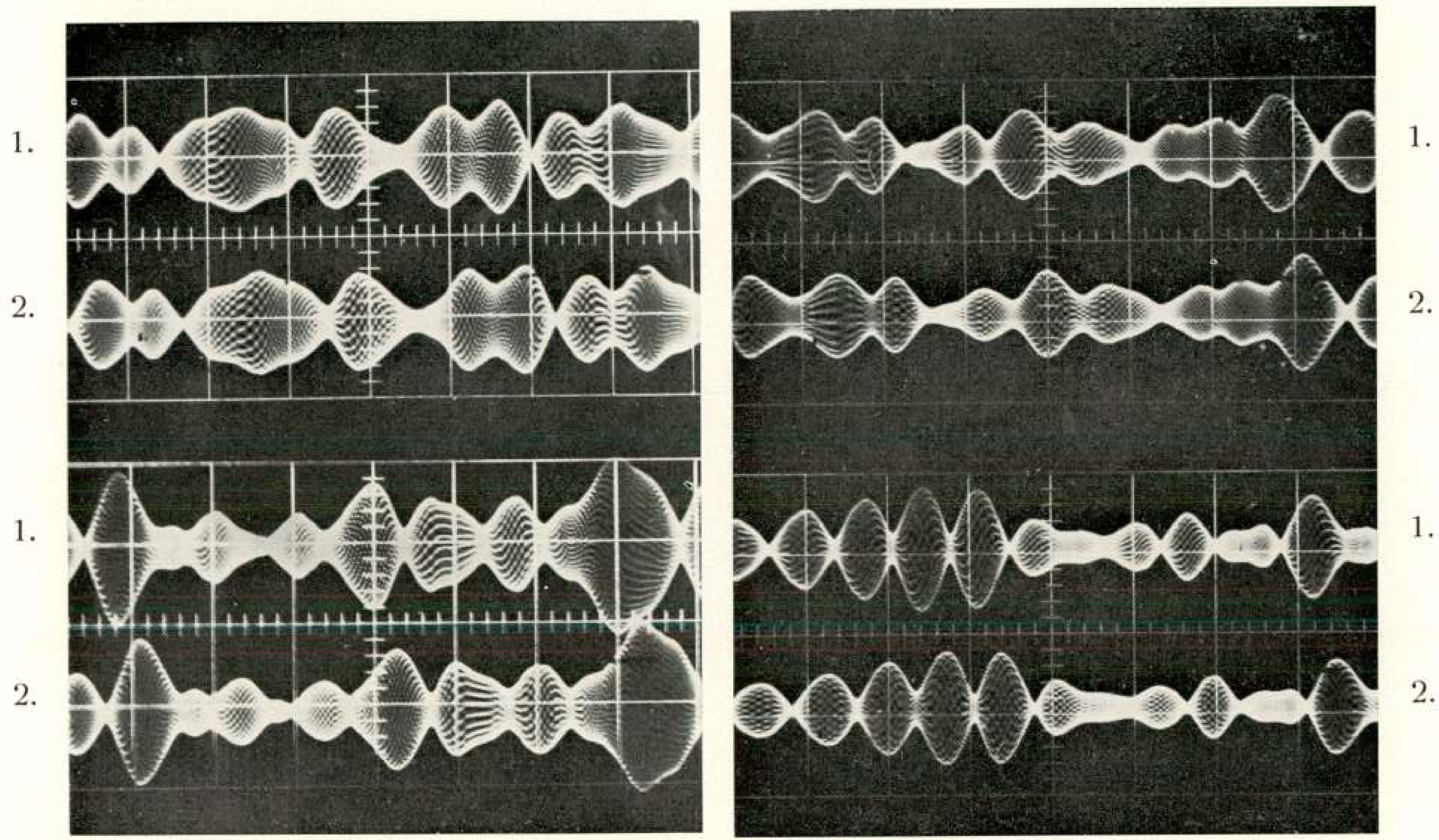



Fig. 7
Signal reconstruction;
trace 1: original signal; trace 2: reconstructed signal.
$F = 760$ samples/sec. Time scale 10 ms/cm.

1.
2.

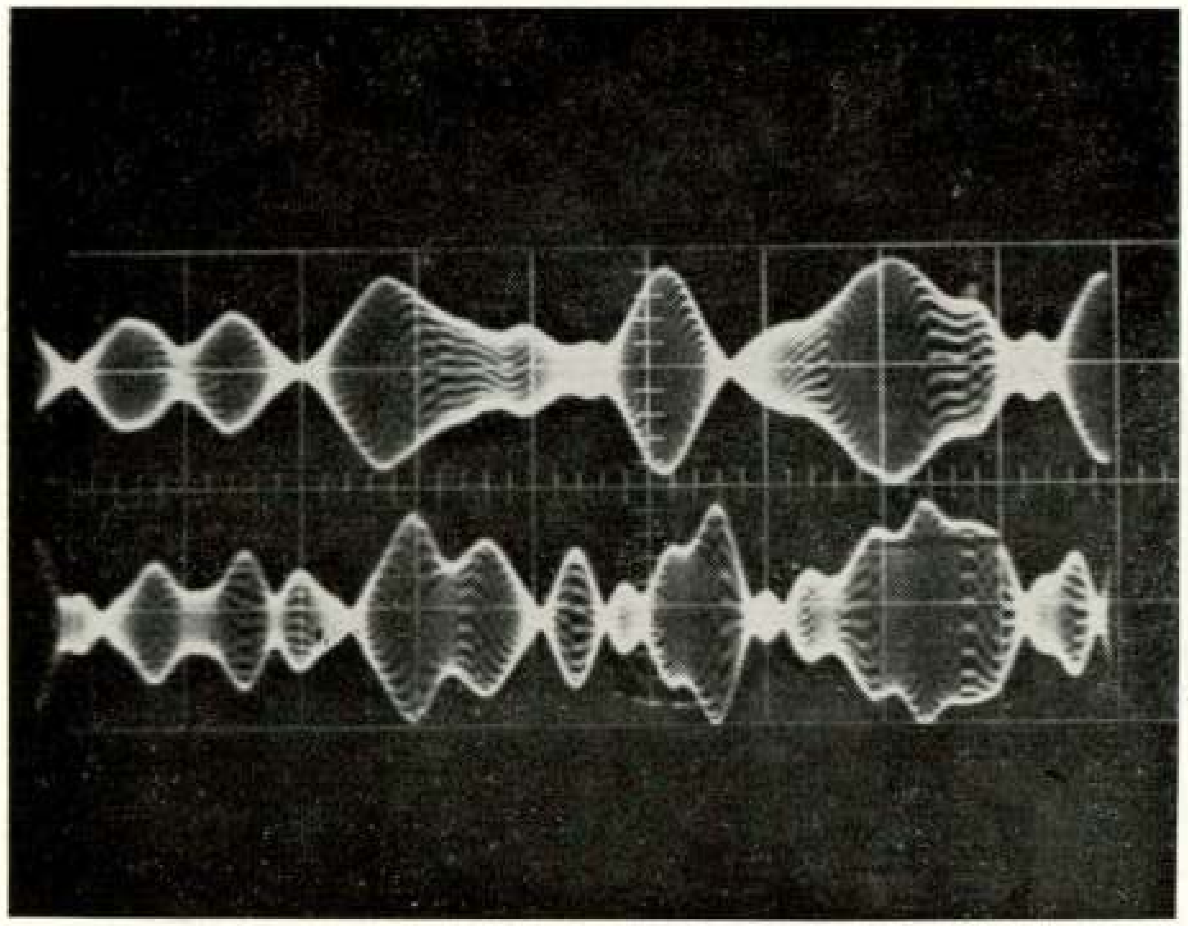
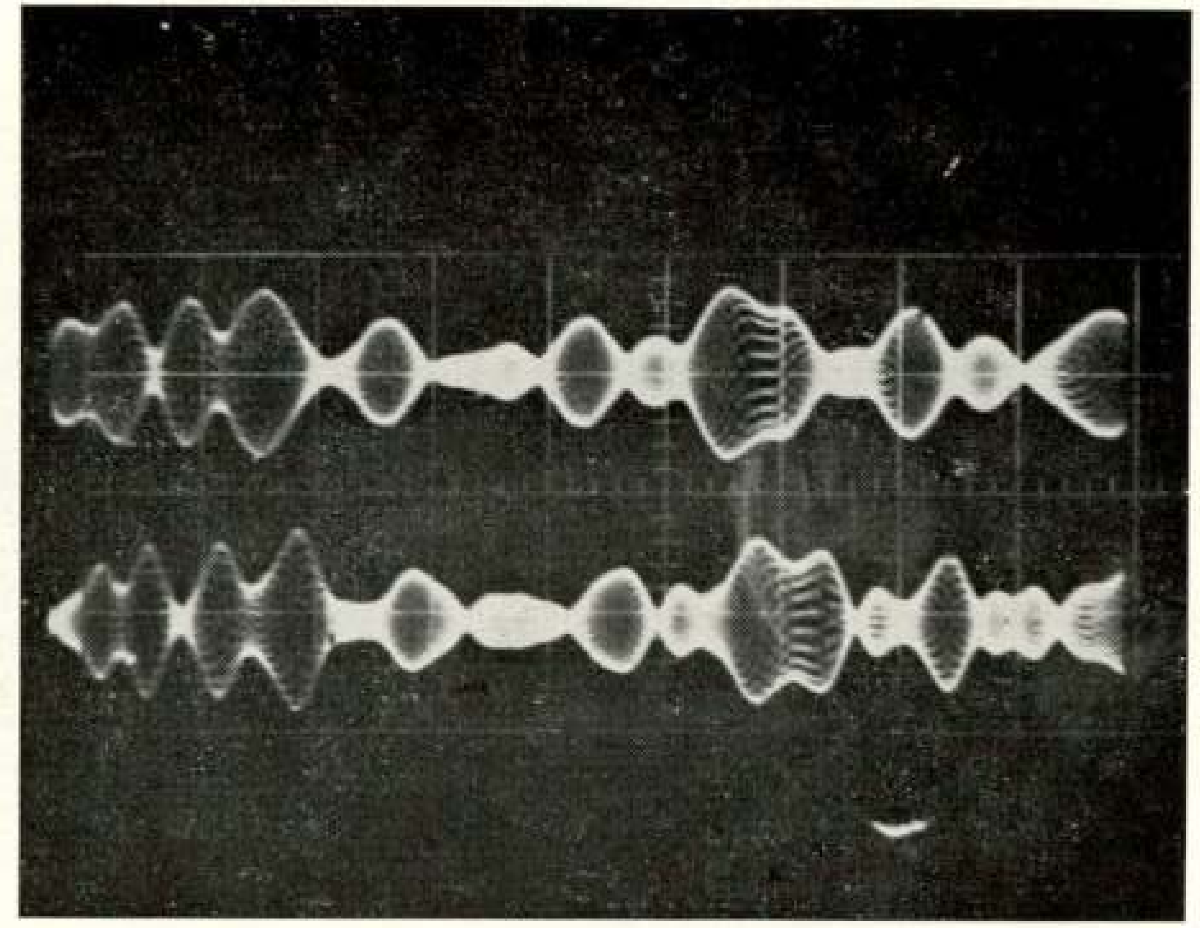


Fig. 8



1.
2.

Fig. 9

Figs. 8, 9, Signal reconstruction;
trace 1: original signal;
trace 2: reconstructed signal;
Time scale 10 ms/cm.
Fig. 8: $F = 748$; Fig. 9: $F = 752$

1. 2.

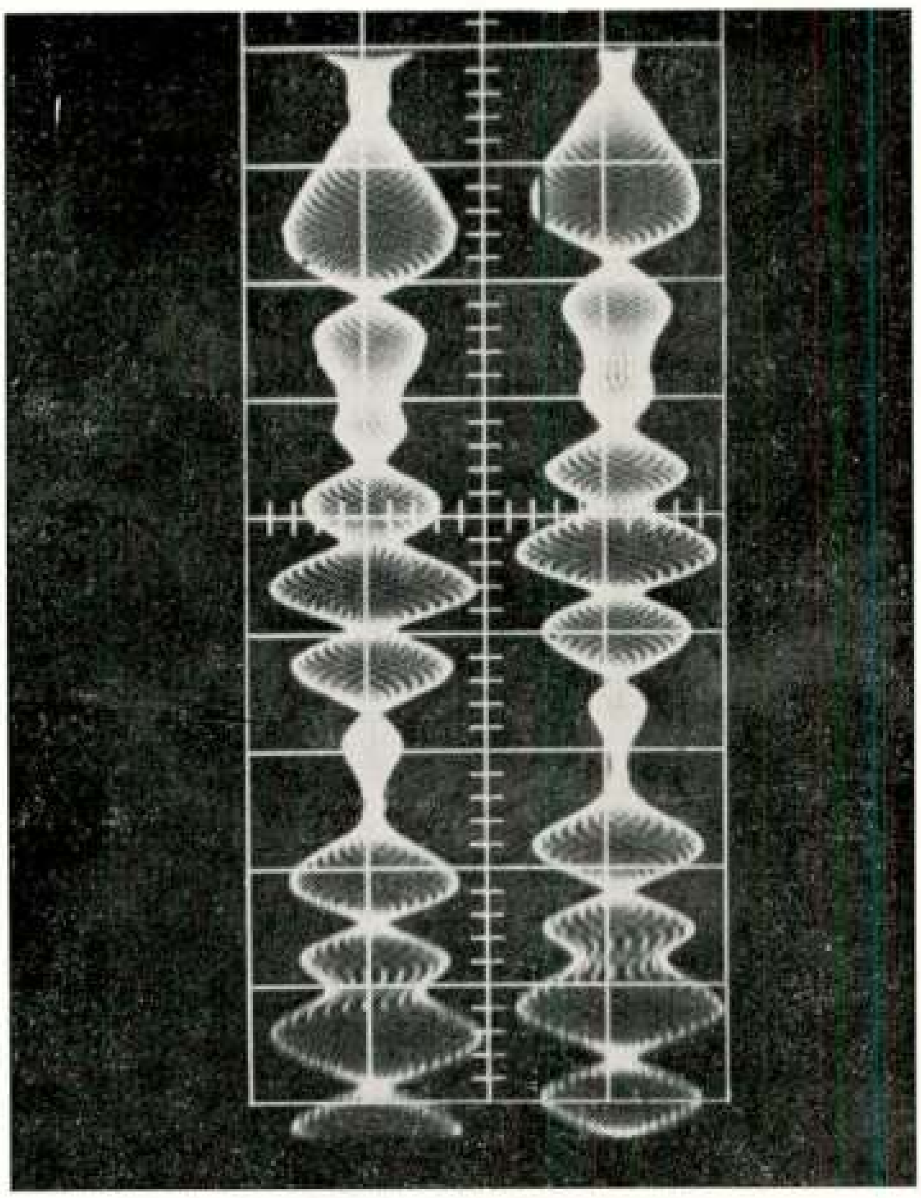


Fig. 11

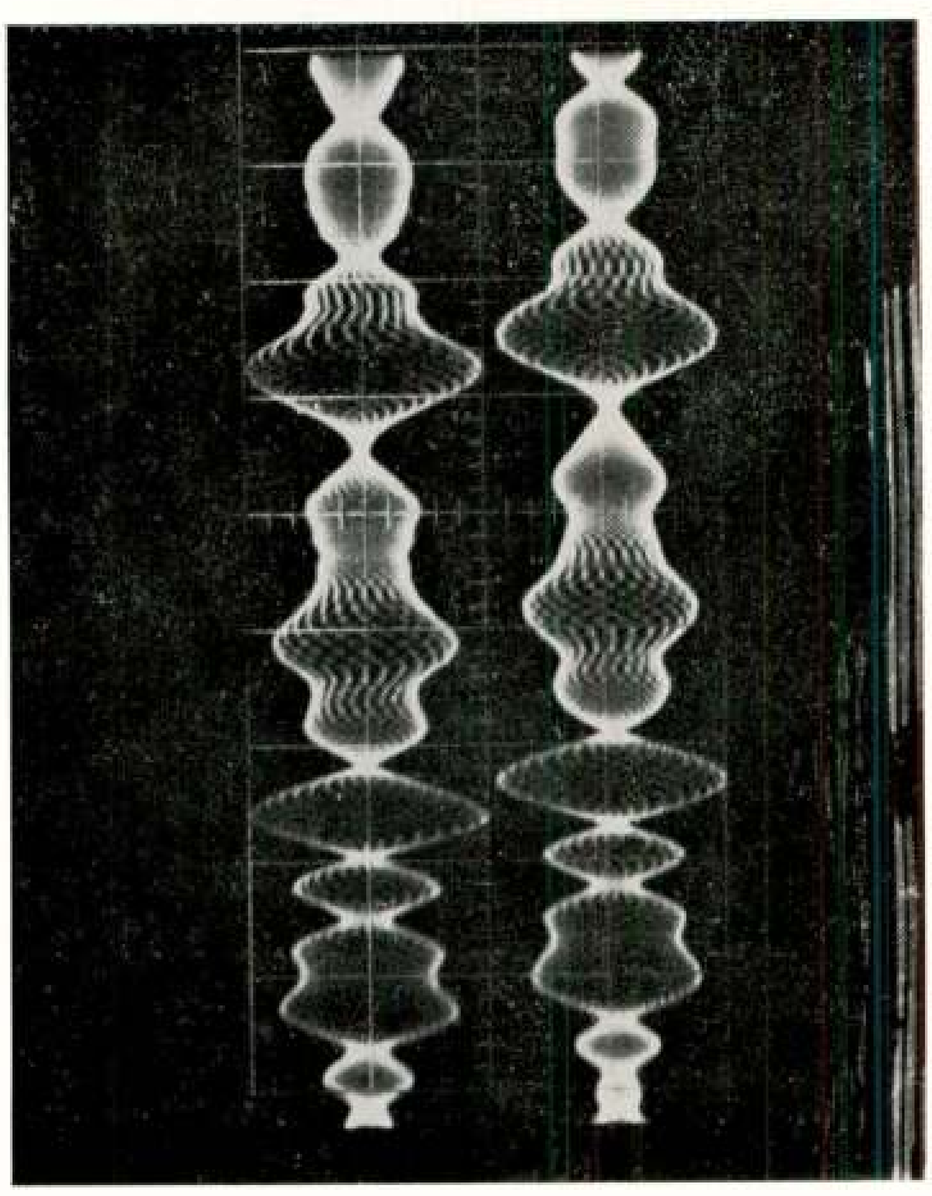


Fig. 10

1. 2.

Figs. 10, 11. Signal reconstruction;
trace 1: original signal;
trace 2: reconstructed signal.
Time scale 10 ms/cm.
Fig. 10: $F = 756$; Fig. 11: $F = 760$

1.
2.

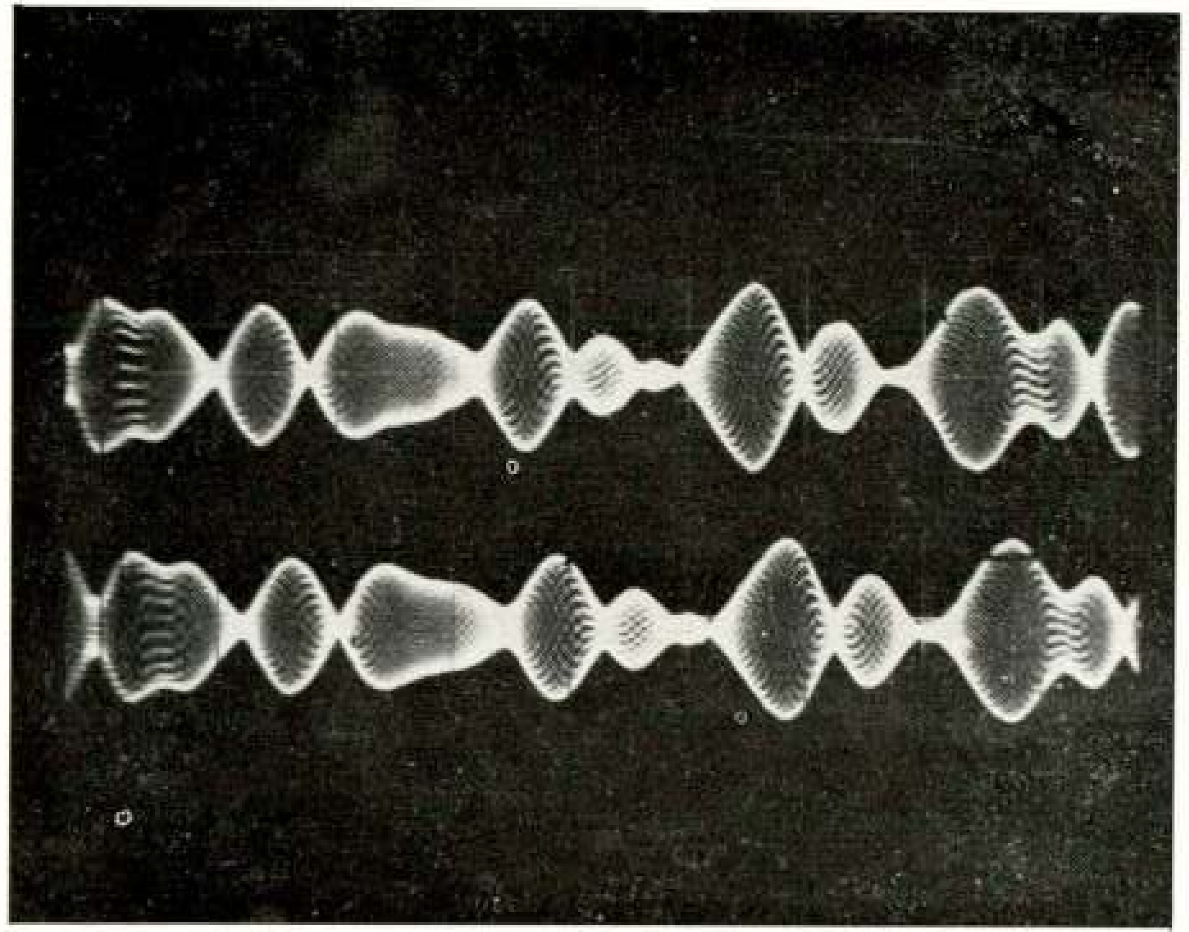
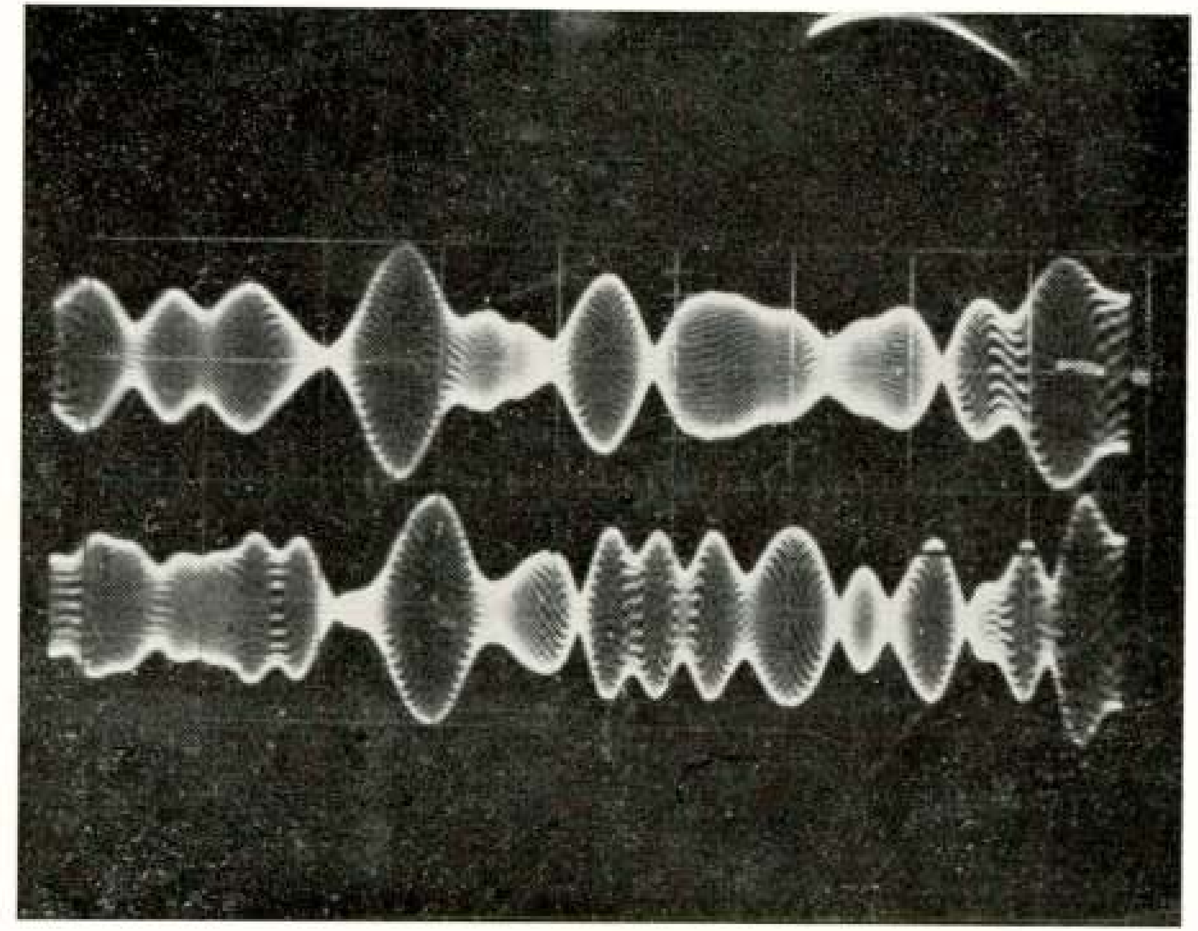


Fig. 12



1.
2.

Fig. 13

Figs. 12,13. Signal reconstruction;
trace 1: original signal;
trace 2: reconstructed signal.
Time scale 10 ms/cm.
Fig. 12: $F = 764$; Fig. 13: $F = 768$

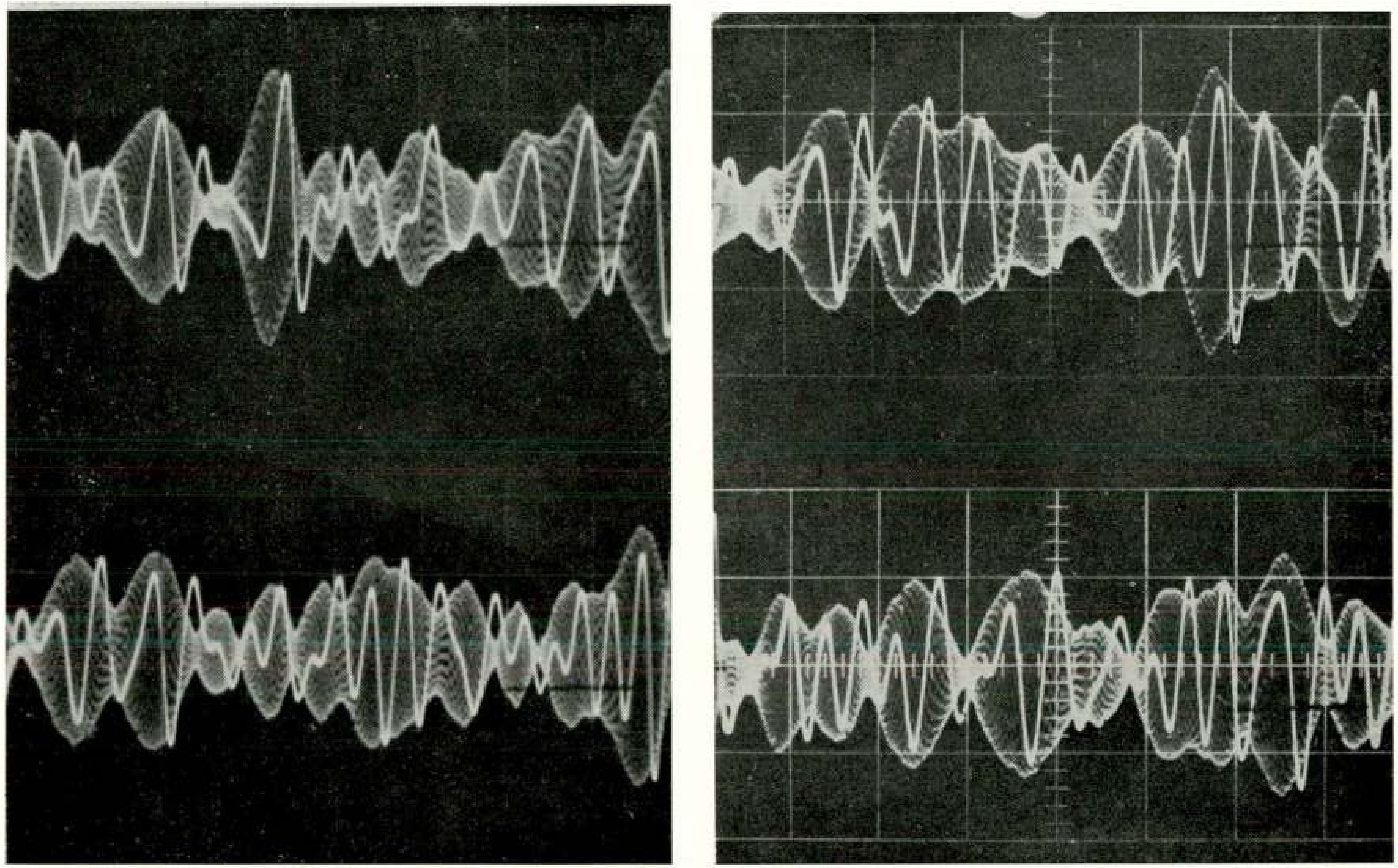


Fig. 14
Reconstruction of the low frequency component of the sample sequence.
$F = 820$ samples/sec. Time scale 10 ms/cm.

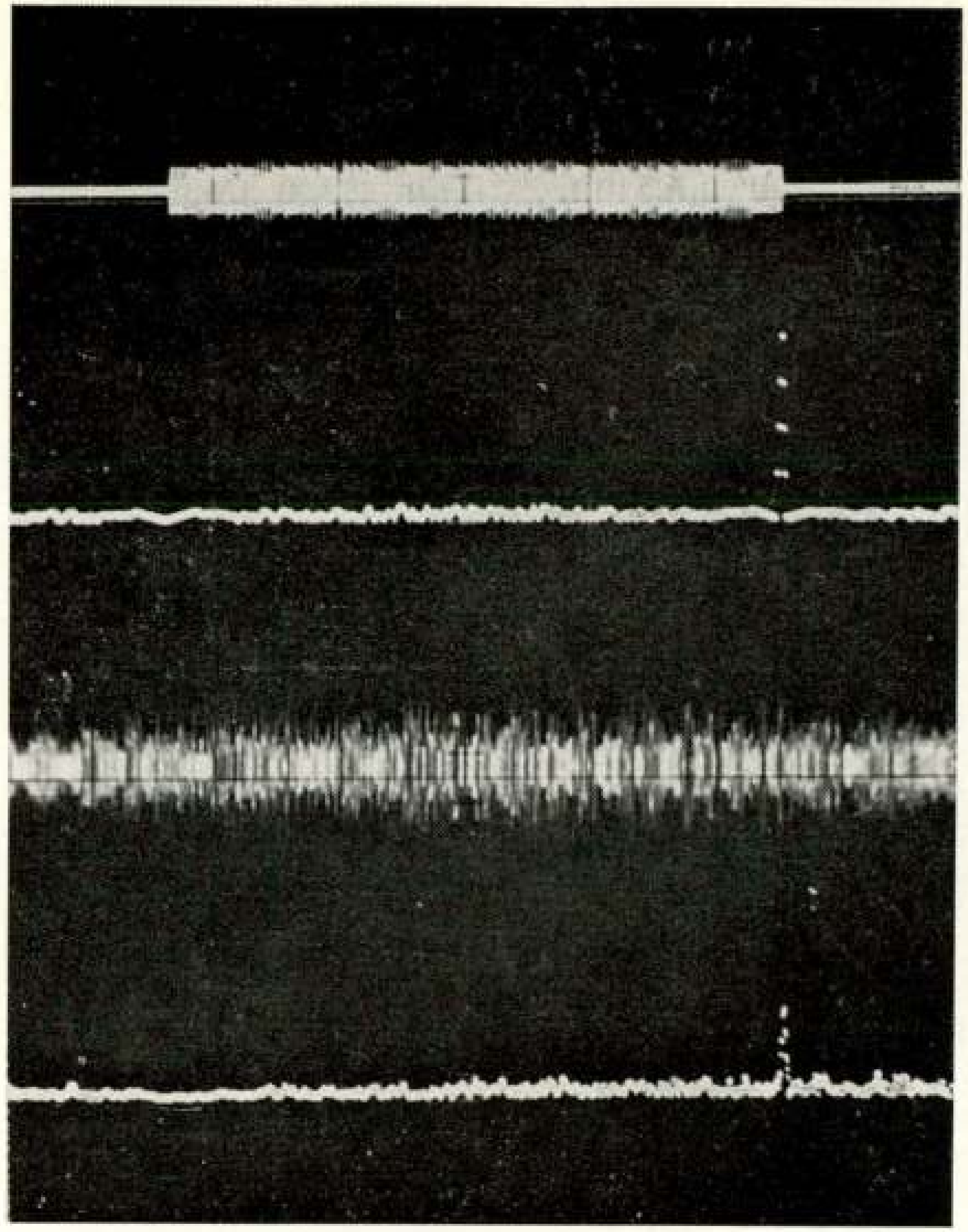



Fig. 15
Sampling correlator;
trace 1: modulated input pulse,
trace 2: output peak,
trace 3: same input pulse in noise,
trace 4: output peak.

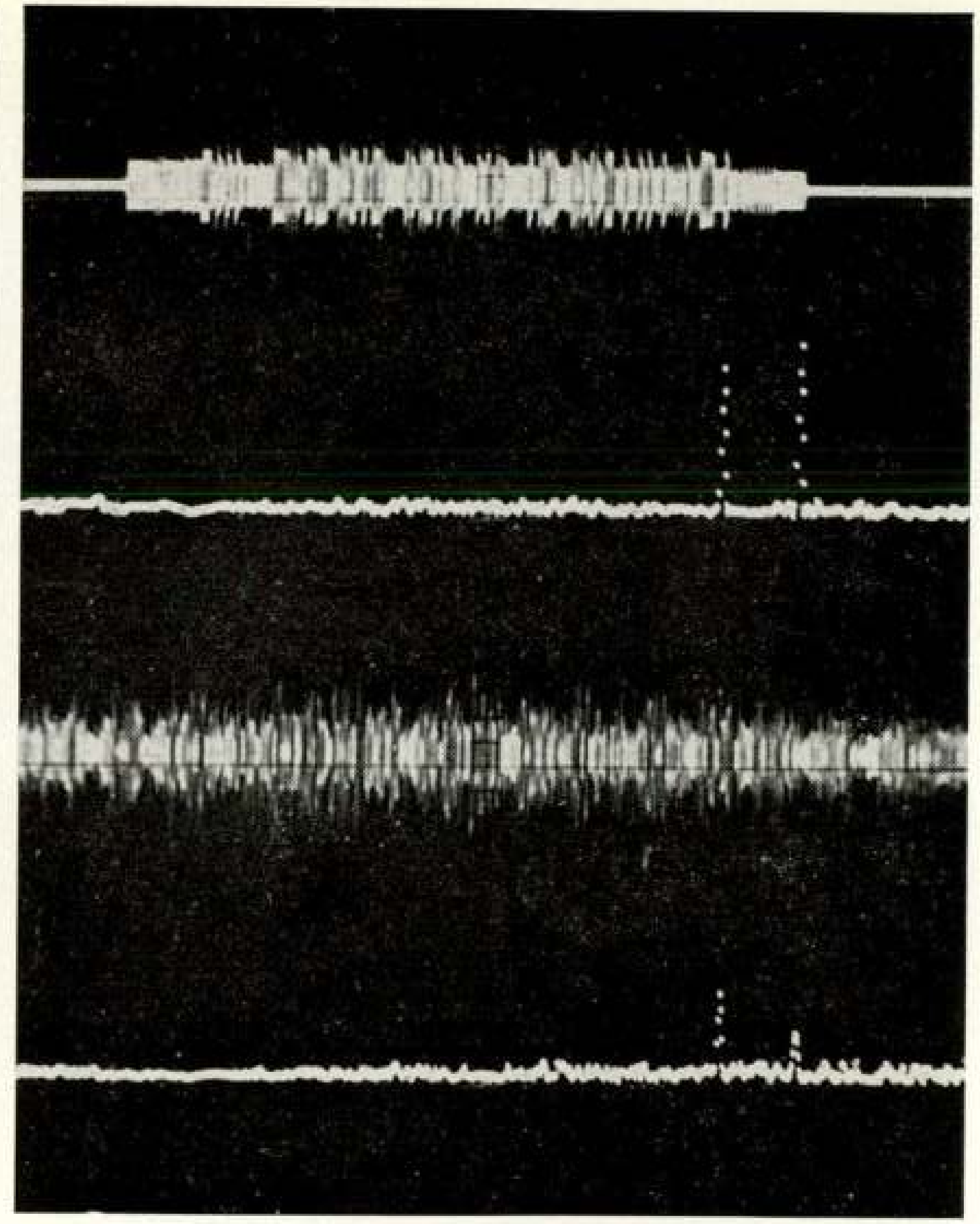



Fig. 16:
Sampling correlator;
trace 1: two overlapping input pulses,
trace 2: corresponding output peaks,
trace 3: overlapping pulses in noise,
trace 4: output peaks.

the lowest spectral component of $S'(f)$ was filtered out with a lowpass filter. The reconstructed signal is written inside the original one to facilitate a comparison between the two. Note that the carrier frequency has disappeared completely. Only the modulation is left, as may be expected in this particular case. As in the previous photographs, the output signal is delayed with respect to the input signal. Fig. 14 was taken at a sampling rate $F = 820$ samples/second.

## 6. *Theoretical and practical applications*
### *The sifting-property*

The applications of sampling theorems are to be found on two different levels.

In the first category are hardware applications, as explained briefly in the introduction, where one is forced by technological arguments to operate on a series of samples, rather than on a continuous signal.

A second type of application is the use of sampling theorems to facilitate the solution of theoretical problems. A problem concerning continuous signals is translated into sample language, solved, and translated back into continuous language.

For this second type of application, some caution in using the present sampling method should be recommended.

1. If we choose a number of arbitrary sample values and construct a signal from these using eq.(14), we can not expect the result to be a narrowband signal satisfying eq.(2). This is because the elementary functions generally occupy a bandwidth larger than $B$, unless we choose a rectangular transfer function for the reconstruction process:

$$\left.\begin{aligned} H(f) &= 1 \text{ if } f_0 - \tfrac{1}{2} B < |f| < f_0 + \tfrac{1}{2} B\,, \\ &= 0 \text{ elsewhere.} \end{aligned}\right\} \tag{20}$$

2. The elementary functions are not necessarily mutually orthogonal, in contradistinction to Shannon's and Woodward's sampling theorems (eqs. (4), (5)). In general, we have

$$\int e(t-t_k)e^*(t-t_l)\mathrm{d}t \neq 0 \text{ for all } k,\ l\,. \tag{21}$$

(This does not imply, however, that a specific orthogonal solution for the $e$-functions never exists.)

3. Generally, the following property holds:

$$\left.\begin{aligned} e(t) &\neq 0 \quad \text{for } t = t_k;\ \ k = \pm 1,\ \pm 2,\ \pm 3, \ldots\ldots; \\ &\neq 1 \quad \text{for } t = 0\,. \end{aligned}\right\} \tag{22}$$

Again, this is in contrast with the elementary functions of eqs.(4) and (5).

For these reasons, there is no advantage in using the theorem of eq. (14) for solving a theoretical problem. In this type of problems the sampling rate plays no role and one of the theorems of section 2 is to be preferred because of their elegant mathematical properties.

In practical applications, the situation is reversed. The sampling rate has to be kept as low as possible, whereas the properties of the elementary functions are only of minor concern. Here, eq.(14) offers a distinct advantage. In this context, however, one final point remains to be settled.

Up to now we have chiefly been interested in the problem of reconstructing the signal from its samples, as this is one of the essential features of any sampling method. As pointed out in the introduction, however, most of the practical applications consist of operating in a specific way on the sequence of samples in order to reach a desired result. For these applications, an important question is whether the operation on the samples is equivalent to the corresponding operation on the continuous signal.

Let us study this problem for an arbitrary linear operation, for which a general expression can be written as

$$q = \int s(t)\ p(t)\ \mathrm{d}t\,, \tag{23}$$

$s(t)$ is the narrowband signal to be operated upon, and $p(t)$ is the weighting function associated with the linear operation. Without loss of generality $p(t)$ can be assumed to possess the same narrowband properties as $s(t)$. Hence

$$\left.\begin{aligned} & s(t) = \sum_k s(t_k)\ e(t-t_k) \text{ and } p(t) = \sum_l p(t_l)\ e(t-t_l)\,, \\ & \text{with } t_k = \frac{\mathrm{k}}{F}\ ;\ F \text{ on one of the admitted intervals.} \end{aligned}\right\} \tag{24}$$

For sampling theorems based on a system of orthogonal elementary functions, there is no problem. Orthogonality automatically guarantees the desired equivalence. This is nothing but the well known inner product theorem for orthogonal expansions. Assuming for a moment that

$$\left.\begin{aligned} \int e(t-t_k)\ e(t-t_l)\,\mathrm{d}t &= c \text{ if } \mathrm{k} = l, \\ &= \text{ if } \mathrm{k} \neq 1, \end{aligned}\quad \text{(orthogonality)}\right\} \tag{25}$$

it follows that

$$q = \int s(t)\ p(t)\mathrm{d}t = \sum_k \sum_l s(t_k)\ p(t_l) \int e(t-t_k)\ e(t-t_l)\mathrm{d}t =$$
$$= c \sum_k s(t_k)\ p(t_k)\ . \qquad (26)$$

Hence the continuous and sampled operations are equivalent. The sampling theorems of section 2 are systems based on orthogonal elementary functions.

However, as pointed out above, the $e(t)$ functions of eq.(14) are not orthogonal in general. Nevertheless, relation (26) remains valid. This is because the elementary functions possess the „sifting property" with repect to the narrowband signals defined by eq.2:

$$\int s\ (t)\ e(t-\tau)\ \mathrm{d}t = \frac{1}{F}\ s(\tau)\ ;\ \text{sifting property}). \qquad (27)$$

The same property holds for $p(t)$. A proof is given in the appendix. Eq.(27) is called the sifting property because it sifts out the signal value at any desired time instant. In this respect $e(t)$ behaves in the same way as a Dirac delta function:

$$\int s(t)\,\delta(t-\tau)\ \mathrm{d}t = s(\tau). \qquad (28)$$

Substitution of eq.(24) into eq.(23) yields the desired result:

$$q = \int s(t)\ p(t)\ \mathrm{d}t = \sum_k s(t_k) \int p(t)\ e(t-t_k)\ \mathrm{d}t\,.$$

Application of eq.(27) with $p(t)$ in stead of $s(t)$ gives:

$$q = \int s(t)\ p(t)\ \mathrm{d}t = \frac{1}{F} \sum_k s(t_k)\ p(t_k)\ . \qquad (29)$$

Hence the operation on the signal samples yields exactly the same result as the original operation on the continuous signal.

For a number of years, correlators based on the sampling method of eq.(14) have been built at the Physics Laboratory NDRO-TNO. Their main application is by way of matched filters for the detection of modulated sonar signals. A few examples are illustrated in photographs 15 and 16. Traces 1 and 2 of fig. 15 show a phase modulated carrier input pulse and the corresponding correlator output, respectively. The same is shown on traces 3 and 4; here the input pulse is disturbed by additive noise of the same bandwidth as the pulse. Fig. 16 shows the results when two overlapping input pulses are present. Traces 1 and 2: noise absent; traces 3 and 4: noise present. As the equipment operates on samples of the input signal, the output is also obtained as a series of samples. This is clearly observed in the output peaks of fig. 15 and

16. If desired, a continuous output signal can be obtained from these samples using the technique described in section 3.

7. *Conclusions.*

A sampling theorem for narrowband signals has been described, featuring some properties which are of particular importance for practical applications; the samples are equidistant, the sampling rate $F$ is close to the theoretical minimum value, and only the narrowband signal itself has to be sampled.

Hence there is no need for heterodyning the signal down to low frequencies and applying Shannon's wideband theorem, or for sampling both the signal and its Hilbert transform.

The signal can be reconstructed from its samples by feeding them into a suitably designed bandpass filter, centered around the original center frequency. If desired, the signal can be reconstructed at a number of other center frequencies, separated by multiples of $F_0$. Some experimental results were presented.

In signal processing applications, a specific operation has to be performed on the continuous narrowband signal. For linear operations, it was shown that an equivalent operation on the samples yields exactly the same result. This property follows from the fact that the elementary functions have the sifting property with respect to the narrowband signals.

The narrowband sampling theorem is oriented primarily towards practical applications. It is not advised to use the theorem as an aid in solving theoretical problems as it lacks some of the mathematical properties, inherent to other sampling theorems.

APPENDIX

*Proof of sifting property (eq. (27)).*

We have:

$$s(t) \doteq S(f) \quad \textit{and } e((t-\tau) \doteq E(f) \exp -2\pi\, j\, f\, \tau\,. \tag{30}$$

Thus, application of Parseval's theorem gives:

$$\int s(t)\, e(t-\tau)\, dt = \int S(f)\, E(-f) \exp 2\pi\, j\, f \tau\, df\,. \tag{31}$$

According to section 2, eqs. (10), (13), $E(f) = \dfrac{1}{F}$ over the signal bandwidth, both for positive and negative frequencies. Therefore, eq.(31)

simplifies to:

$$\int s(t)\, e(t-\tau)\, \mathrm{d}t = \frac{1}{F}\int S(f) \exp 2\pi \mathrm{j} f \tau \,\mathrm{d} f = \frac{1}{F}\, s(\tau)\,, \tag{32}$$

which is nothing but the sifting property.

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*Manuscript ontvangen 15 februari 1967.*

## CONGRESSEN E.D.

### Zomercursus over microgolftechnieken

Van 11 tot 22 september 1967 wordt aan de Katholieke Universiteit te Leuven, door het Laboratorium van Elektronica, afdeling Microgolftechnieken, een zomercursus georganiseerd.

De cursus richt zich vooral tot hen, die de basiskennis, noodzakelijk voor studie en gebruik van microgolftechnieken, wensen te verwerven en te verdiepen. Een reeks seminaries, oefen- en herhalingszittingen worden georganiseerd; bovendien kunnen de deelnemers praktische proeven op het microgolflaboratorium uitvoeren; deze proeven lichten de cursus toe en geven een inzicht in de diverse meettechnieken.

De deelnemers ontvangen gedrukte nota's over de cursus en het laboratorium, evenals teksten, die de seminaries en oefeningen begeleiden. Het aantal deelnemers is echter beperkt door het aantal beschikbare plaatsen op het laboratorium.

De cursus behandelt de volgende onderwerpen: Overzicht van het elektromagnetisme - Transmissielijnen - Reflectie- en verstrooiingsmatrix - Geleide golven - Hindernissen in golfgeleiders - Trilholtes - Juncties - Passieve niet-reciproke elementen - Periodieke structuren en filters. Seminaries: Meettechnieken - Microgolfbuizen - Ferrieten - Elementen met lage ruis; toepassing op ruimtelijke communicatie - Microgolfhalfgeleiders - Industriële toepassingen.

Het ligt in de bedoeling van de organisatoren een intense wederzijdse samenwerking te verwezenlijken. De cursus staat onder de leiding van Prof. A. van der Vorst, Dr. sc. appl., M.S.E.E. (M.I.T. - U.S.A.).

Inschrijvingen worden verwacht vóór 15 juli 1967. Het inschrijvingsrecht bedraagt 10.000 Bfr; hierin zijn middagmalen en verfrissingen begrepen. Nadere inlichtingen worden verstrekt door het Laboratorium voor Elektronica, Kardinaal Mercierlaan 94, Heverlee.

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## WETENSCHAPPELIJK ONDERWIJS

### Technische Hogeschool Eindhoven

Bij Koninklijk Besluit van 30 maart 1967 is *Dr.-Ing. H. J. Butterweck*, medewerker aan het Natuurkundig Laboratorium der N.V. Philips' Gloeilampenfabrieken te Eindhoven, benoemd tot gewoon hoogleraar in de theoretische elektrotechniek aan de Technische Hogeschool te Eindhoven.

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## VARIA

### „Seminar" over telegrafie

In het kader van de ontwikkelingshulp organiseerde PTT een „seminar" over telegrafie, van 17 tot 28 april 1967.

Er hadden zich voor deze in Den Haag gehouden conferentie 18 deelnemers gemeld, afkomstig uit 13 landen, nl. Dahomey, Ethiopië, Gambia, Ghana, Indonesië, Jordanië, Kenya, Koeweit, Liberië, Tunesië, Turkije, Venezuela en, als enig geïndustrialiseerd land, Ierland.

Het initiatief tot het houden van dergelijke bijeenkomsten, die een wezenlijke bijdrage vormen tot de ontwikkelingshulp, werd in 1961 genomen door de Internationale Telecommunicatie Unie (I.T.U.), een van de gespecialiseerde organen van de Verenigde Naties. Jaarlijks worden in de wereld twee tot drie seminars gehouden, waarbij er naar wordt gestreefd het gehele terrein van de telecommunicatie te bestrijken. Voor het eerst stond nu als hoofdthema de telegrafie op het programma.

Alle deelnemers aan de Haagse conferentie hebben in hun dagelijks werk met de telegrafie te maken, als regel met de technische kant. Het is de bedoeling, dat zij, naast enige praktische kennis, ook ideeën opdoen, die hun gezichtsveld verruimen en kunnen bijdragen aan de ontwikkeling van de telegraafdienst in hun land. Een typisch bewijs van de voortschrijdende emancipatie is het feit, dat de beide deelnemers uit Turkije vrouwelijke ingenieurs waren.

## Genormaliseerde elektrotechnische symbolen

Het Nederlands Normalisatie-Instituut heeft een tweede gewijzigde druk gepubliceerd van het boekje GES A6 in agendaformaat (14,7 × 10,3 cm), bevattende de genormaliseerde elektrotechnische symbolen.

Dit handige boekje bevat o.a. 194 symbolen voor elektrotechnische tekeningen. De tekst en de tekeningen zijn zeer duidelijk en overzichtelijk. Het is alleen jammer, dat de twee er in voorkomende verkleind afgedrukte normbladen (NEN 333 en V 1382) zo klein zijn uitgevallen, dat men wel zeer goede ogen moet hebben om deze te kunnen lezen.

Het 16 bladzijden tellende boekje is voor het geringe bedrag van *f* 0,50 (bij meer dan 10 exemplaren korting) verkrijgbaar bij het Nederlands Normalisatie-Instituut, Polakweg 5, Rijswijk (Z.H.).

## „Intelsat II"

Op 24 maart 1967 is voor de „Communication Satellite Corporation" (Comsat) de satelliet „Intelsat II" (fig. 1) gelanceerd, bestemd voor communicatie tussen de delen der aarde, in fig. 2 aangegeven. Evenals de in april 1965 gelanceerde „Early Bird" bevindt de nieuwe satelliet zich boven de equator; Early Bird op 29° en Intelsat II op 5° Westerlengte. Eenzelfde satelliet was gelanceerd op 11 januari 1967 en bevindt zich boven de Pacific, voor communicatie tussen USA en Japan.

Het grondstation op de Canarische Eilanden, ontwikkeld en gebouwd door ITT Federal Laboratories, zal tevens de verbinding verzorgen ten behoeve van het Apollo „Man-op-de-Maan"-project, waarbij het als zendend en ontvangend volgstation dienst zal doen via Intelsat II. Dit station beschikt over dubbel uitgevoerde automatisch volgende parabolische antennes met 14 meter diameter.

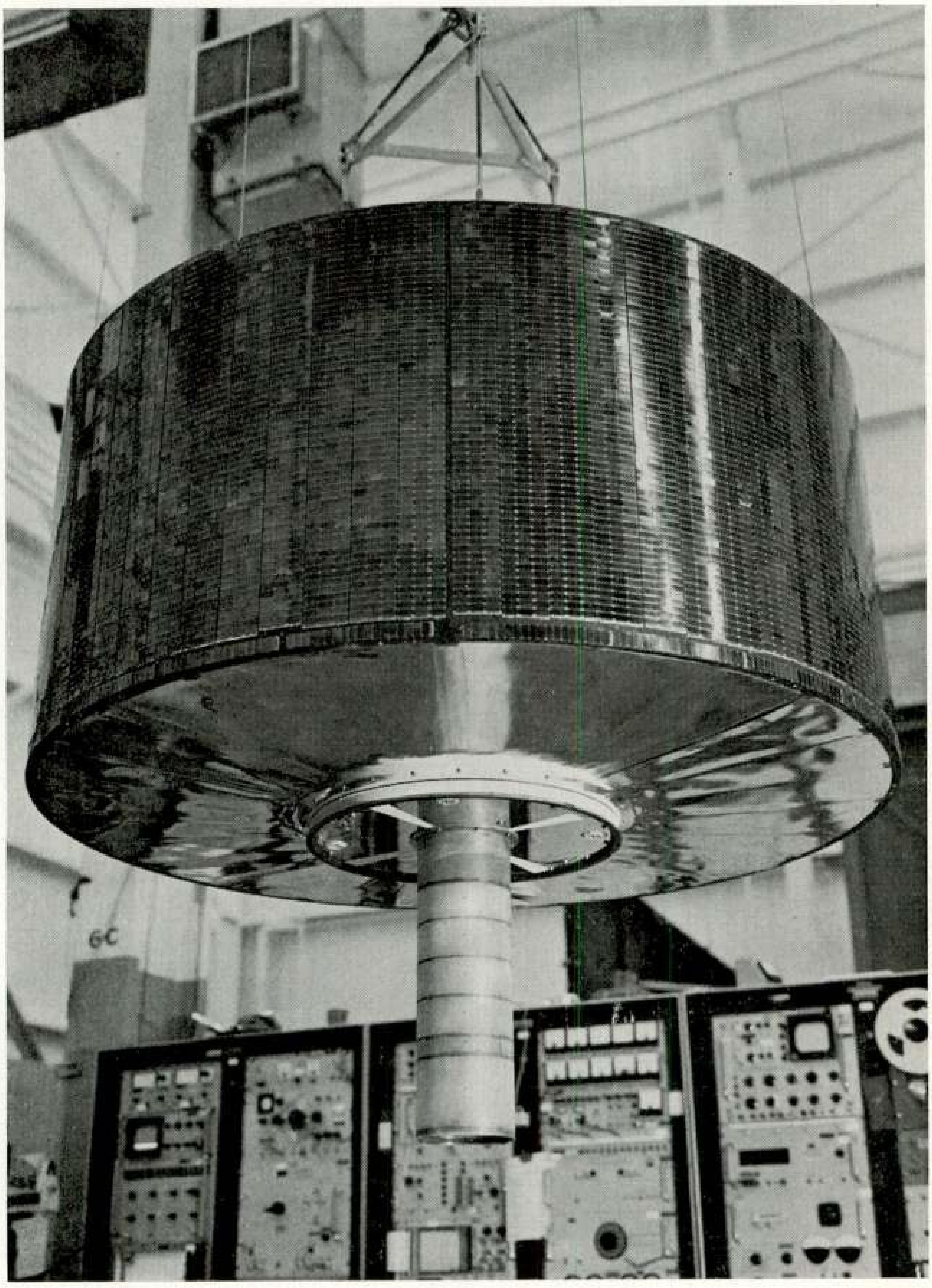


Fig. 1
De satelliet „Intelsat II” tijdens de beproeving in het laboratorium van Hughes Aircraft Company, waar deze satelliet is ontwikkeld

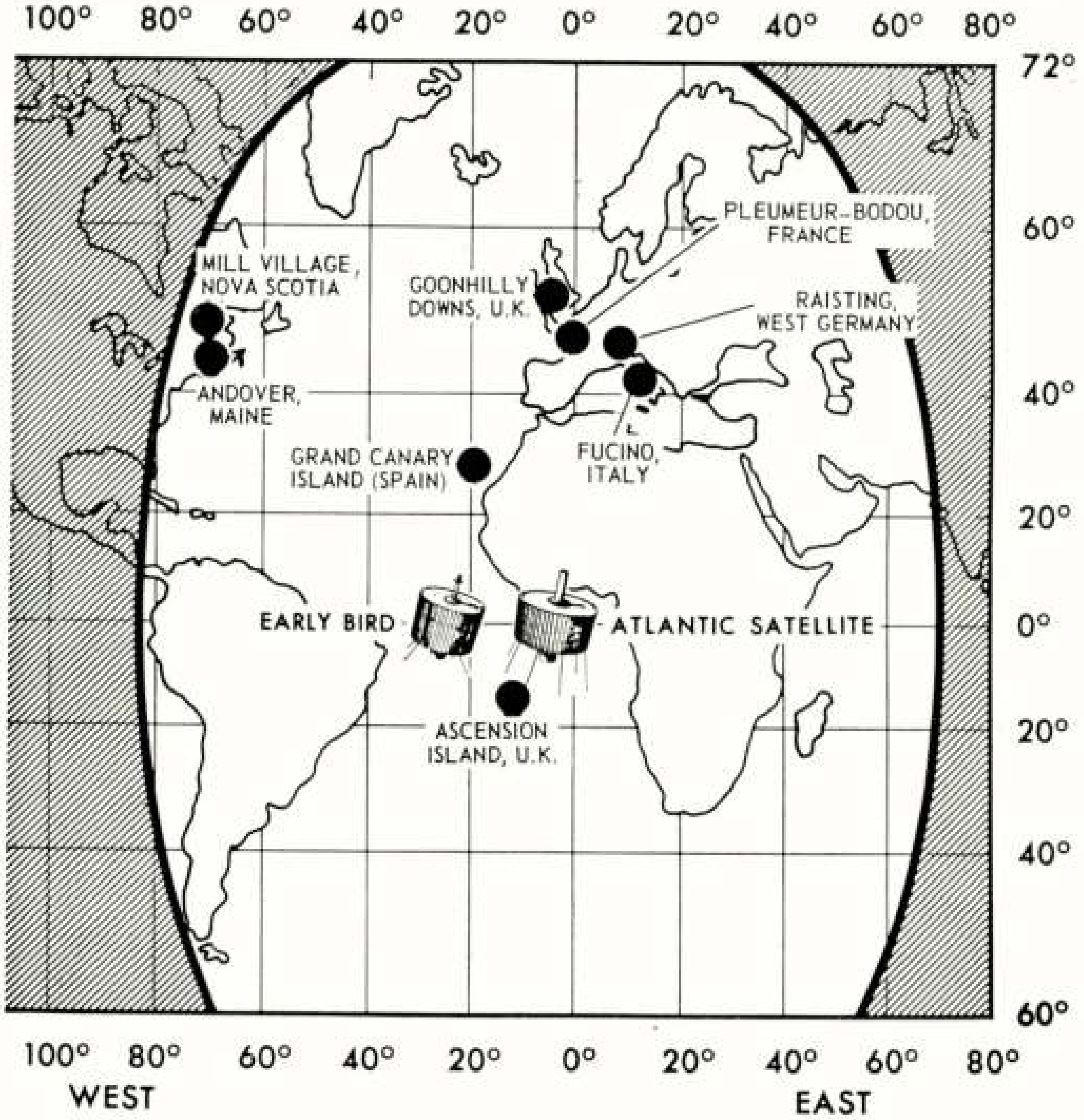


Fig. 2
Plaats der beide satellieten Early Bird en Atlantic Satellite Intelsat II. Het niet-gearceerde deel der aarde is zichtbaar vanuit laatstgenoemde satelliet. De stations, waartussen communicatie plaats vindt via deze satelliet, zijn aangegeven

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## BOEKAANKONDIGING

*Microwave breakdown in gases,* door A. D. MacDonald. 201 blz., vele figuren. Uitg. J. Wiley & Sons, Londen. 1966. Prijs 64 s.

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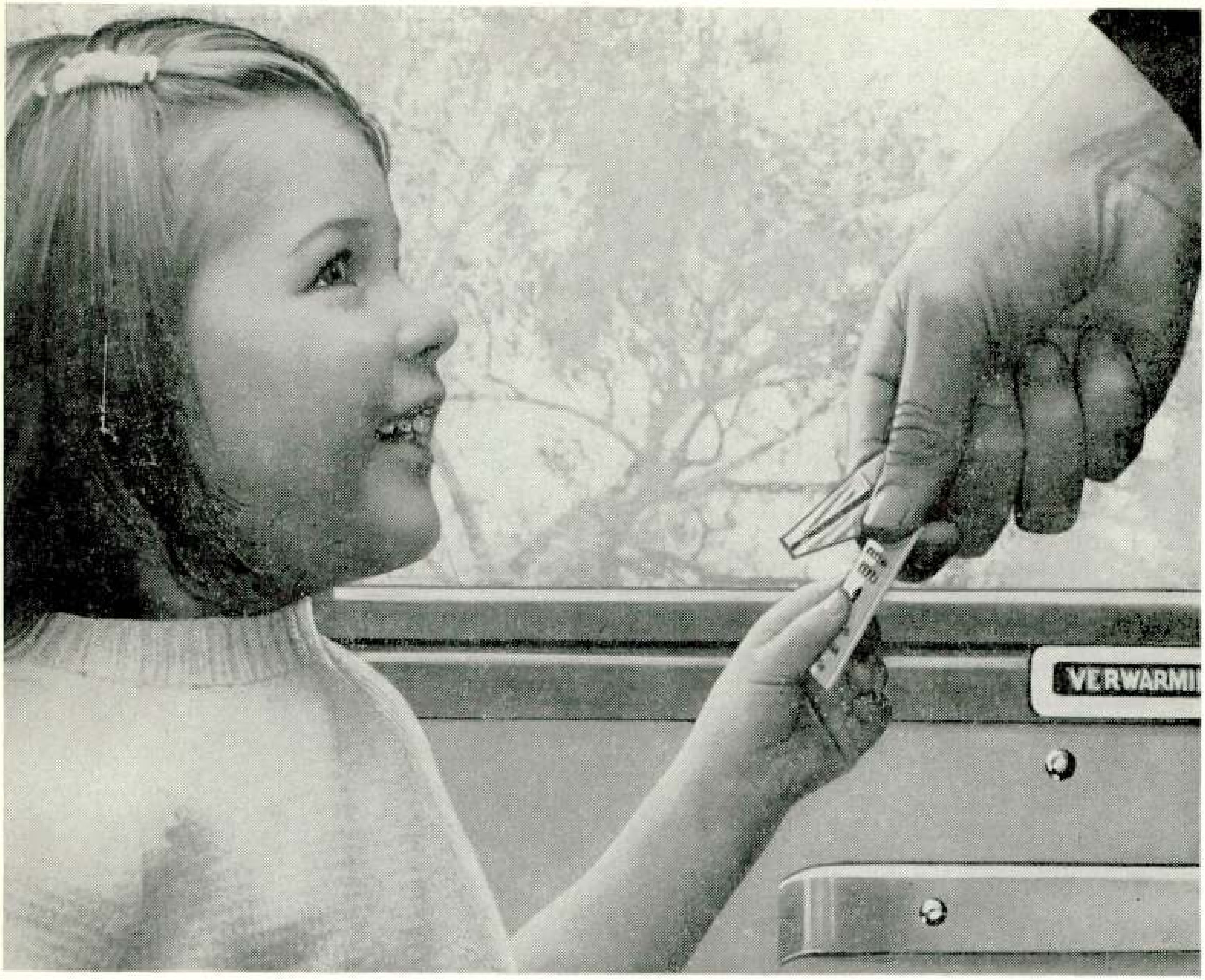
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